AN ADAPTIVE RESIDUAL LOCAL PROJECTION FINITE ELEMENT METHOD FOR THE NAVIER–STOKES EQUATIONS

RODOLFO ARAYA, ABNER H. POZA, AND FRÉDÉRIC VALENTIN

Abstract. This work proposes and analyses an adaptive finite element scheme for the fully non-linear incompressible Navier-Stokes equations. A residual a posteriori error estimator is shown to be effective and reliable with respect to the natural norms. The error estimator relies on a Residual Local Projection (RELP) finite element method for which we prove well-posedness under mild conditions. Several well-established numerical tests assess the theoretical results.

1. Introduction

A posteriori error analysis for adaptive finite element methods has been a very active and successful subject of research since the pioneering work of Babuska and Rheinboldt in [8]. In the context of fluid flow problems, researchers have been focused on improving numerical precision while making the computational cost affordable. For the Stokes problem we cite the relevant works by Verfürth [31], Bank and Welfert [9] and Ainsworth and Oden [2]. Regarding the Navier-Stokes equations it is worth mentioning the residual-based estimators proposed in [7, 13, 17, 22], the goal-oriented scheme in [12], and the hierarchical a posteriori error estimator in [5] and the ones based on local problem solutions in [21, 25] (see also [1, 33] for an overview).

Stabilized finite element methods for Navier-Stokes equations use equal-order pairs of interpolation spaces for the velocity and pressure. Well-balanced numerical diffusion may be also incorporated into such methods through the stabilization parameter. This is a crucial point when it comes to numerically solving advection dominated (high Reynolds number) flows (see [18, 29] or [14], for instance). The association of stabilized methods with a posteriori error estimators greatly improves the quality of the numerical solutions while keeping the computational cost relatively low (see [3]). Such a feature is particularly attractive if one approximates solutions with multiple scales, as in the case of the non-linear Navier-Stokes equations.

Residual Local Projection (RELP) stabilized methods add new stabilization to the Galerkin method as a result of a space enriching strategy. First proposed in [10, 11] for the Stokes operator, and further extended to the fully non-linear Navier-Stokes equations in [4], these methods rely on the solution of element-wise problems. Such a local solution designs the stabilization parameter with the right dose of numerical diffusion and stabilizes the equal order and the simplest elements. In this work, we develop a new residual-based a posteriori error estimator for the non-linear incompressible Navier-Stokes equations. To this end, we consider a variation of the RELP method proposed in [4] for which we prove the existence and the uniqueness of the

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solution. Also, we prove that the new estimator is effective and reliable following closely the theory presented in Verfürth [32].

The paper is organized as follows: Section 2 states the problem and introduces preliminary results. Section 3 presents the RELP method and the proof of the existence of a solution. The residual a posteriori error estimator is analyzed in Section 4, followed by numerical validations in Section 5. Finally, concluding remarks are given in Section 6 and the appendix includes the proof of a local unique solution for the RELP method.

2. Model problem and Preliminary results

The steady incompressible Navier–Stokes problem consists of finding the velocity $\boldsymbol{u}$ and the pressure p solution of

$$\text{(NS)} \qquad \begin{cases} -\nu\,\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\,\boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \qquad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^2$ is a polygonal open domain, $\nu \in \mathbb{R}^+$ is the fluid viscosity and $\boldsymbol{f} \in L^2(\Omega)^2$ is a given function. We set $\mathbf{V} := H_0^1(\Omega)^2$ and $Q := L_0^2(\Omega)$ and introduce the weak form of (NS): *Find* $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ *such that*

$$\nu\,(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + ((\nabla\boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) - (q, \nabla\cdot\boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } (\boldsymbol{v}, q) \in \mathbf{V} \times Q, \tag{1}$$

here $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product, where we use the same notation for vector, or tensor, valued functions.

Problem (1) may be rewrite in a more convenient form in view of analysis. To this end, consider the operator $F : \mathbf{V} \times Q \longrightarrow (\mathbf{V} \times Q)'$ defined by

$$\langle F(\boldsymbol{u}, p), (\boldsymbol{v}, q)\rangle := \nu\,(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + ((\nabla\boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) - (q, \nabla\cdot\boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{v}),$$

where $\langle\cdot,\cdot\rangle$ is the duality product in $(\mathbf{V} \times Q)' \times (\mathbf{V} \times Q)$. Note that (1) is equivalent to: *Find* $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ *such that*

$$\langle F(\boldsymbol{u}, p), (\boldsymbol{v}, q)\rangle = 0 \qquad \forall(\boldsymbol{v}, q) \in \mathbf{V} \times Q. \tag{2}$$

To present the discrete version of (2) and the numerical analysis of it, we need some notations and also some standard technical results. We denote the derivative of F with respect to $(\boldsymbol{u}, q)$ at $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$ by $D_{\boldsymbol{u},p}F(\boldsymbol{v}, q) \in \mathcal{L}(\mathbf{V} \times Q)$, where $\mathcal{L}(\mathbf{V} \times Q)$ stands for the space of bounded linear mappings acting on elements of $\mathbf{V} \times Q$ with values in $\mathbf{V} \times Q$ and equipped with the norm $\|\cdot\|_{\mathcal{L}(\mathbf{V}\times Q)}$ with its usual meaning.

We assume that problem (2) has a solution $(\boldsymbol{u}, p)$ and $D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)$ is an isomorphism from $\mathbf{V} \times Q$ onto $(\mathbf{V} \times Q)'$ (see Section IV.3.1 in [20]). Also, we assume that there is a constant $R_0 > 0$ such that $(\boldsymbol{u}, p)$ is unique in the ball $\mathbb{B}((\boldsymbol{u}, p), R_0)$ (see Section IV.3.2 in [20]). Thereby, the differential operator $D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)$ is Lipschitz continuous at $(\boldsymbol{u}, p)$, i.e.,

$$\gamma := \sup_{(\boldsymbol{v},q)\in\mathbb{B}((\boldsymbol{u},p),R_0)} \frac{\|D_{\boldsymbol{u},p}F(\boldsymbol{v}, q) - D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)\|_{\mathcal{L}((\mathbf{V}\times Q),(\mathbf{V}\times Q)')}}{\|(\boldsymbol{v} - \boldsymbol{u}, q - p)\|_{\mathbf{V}\times Q}} < \infty.$$

We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family of triangulations of Ω into triangles K with boundary ∂K and diameter $h_K := diam(K)$, and $h := \max\{h_K : K \in \mathcal{T}_h\}$. The set of internal edges F reads $\mathcal{E}_h$ and we define $h_F := |F|$. We denote by $\boldsymbol{n}$ the outward normal vector on ∂K; by $[\![\,\cdot\,]\!]_F$ we mean the jump of v over

F. Given $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$, we denote by $\mathcal{N}(K)$ the set of nodes of K, $\mathcal{N}(F)$ the set of nodes of F, and $\mathcal{E}(K)$ the set of edges of K. Also, we define the following neighborhoods:

$$\widetilde{\omega}_K := \bigcup_{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset} K', \quad , \quad \omega_F := \bigcup_{F\in\mathcal{E}(K')} K' \quad , \quad \widetilde{\omega}_F := \bigcup_{\mathcal{N}(F)\cap\mathcal{N}(K')\neq\emptyset} K',$$

and we define

$$\Pi_S(q) := \frac{(q,1)_S}{|S|},$$

for all $q \in L^2(S)$, where $S \subset \mathbb{R}^2$.

The approximate velocity space $\mathbf{V}_h$ is composed of vector–valued piecewise linear continuous functions with zero trace on $\partial\Omega$. For the pressure, the approximate space Q_h is spanned by piecewise polynomial functions of degree $l, (l = 0, 1)$ with zero mean value on Ω. On such spaces, we use the Clément interpolation operator $\mathcal{I}_h : \mathbf{V} \longrightarrow \mathbf{V}_h$ and the operator $\mathcal{J}_h : Q \longrightarrow Q_h$, where $\mathcal{J}_h$ means either the modified Clément interpolation operator in the continuous pressure case ($l = 1$) or the L^2 orthogonal projection onto the constant space ($l = 0$). Such operators have the following approximability properties (see [15], [16] for details):

$$|\boldsymbol{v} - \mathcal{I}_h\boldsymbol{v}|_{m,K} \leq C\, h_K^{l-m}\, |\boldsymbol{v}|_{l,\widetilde{\omega}_K} \qquad \forall \boldsymbol{v} \in H^l(\widetilde{\omega}_K)^2, \tag{3}$$

$$|\mathcal{I}_h\boldsymbol{v}|_{1,K} \leq C\, |\boldsymbol{v}|_{1,\widetilde{\omega}_K} \qquad \forall \boldsymbol{v} \in H^1(\widetilde{\omega}_K)^2, \tag{4}$$

$$\|\boldsymbol{v} - \mathcal{I}_h\boldsymbol{v}\|_{0,F} \leq C\, h_F^{l-1/2}\, \|\boldsymbol{v}\|_{l,\widetilde{\omega}_F} \qquad \forall \boldsymbol{v} \in H^l(\widetilde{\omega}_F)^2, \tag{5}$$

$$|p - \mathcal{J}_h p|_{i,K} \leq C h_K^{j-i} |p|_{j,\widetilde{\omega}_F} \qquad \forall p \in H^j(\widetilde{\omega}_K), \tag{6}$$

$$\|p - \mathcal{J}_h p\|_{0,F} \leq C h_F^{j-1/2} \|p\|_{j,\widetilde{\omega}_F} \qquad \forall p \in H^j(\widetilde{\omega}_F), \tag{7}$$

where $0 \leq m \leq 2$, $\max\{m, 1\} \leq l \leq k + 1$, and $0 \leq i \leq 1$, $1 \leq j \leq k$. Here and forth, the positive constants C are independent of h but can assume different values in each occurrence.

We equip the space $\mathbf{V} \times Q$ with the following product norm

$$\|(\boldsymbol{v}, q)\| := \left\{\nu\, |\boldsymbol{v}|^2_{1,\Omega} + \frac{1}{\nu}\, \|q\|^2_{0,\Omega}\right\}^{1/2}.$$

Next, we recall some standard results which will be extensively used in the sequel.

Lemma 1. *Given $\boldsymbol{v} \in H^1(K)^2$ it holds,*

$$\|\boldsymbol{v}\|^2_{0,\partial K} \leq C \left\{h_K^{-1}\, \|\boldsymbol{v}\|^2_{0,K} + h_K\, |\boldsymbol{v}|^2_{1,K}\right\}. \tag{8}$$

Proof. See [28] for details. □

Lemma 2. *Given $\boldsymbol{v}_h \in \mathbf{V}_h$ and $p_h \in Q_h$, it holds*

$$\|\boldsymbol{v}_h\|_{\infty,K} \leq C\, h_K^{-1}\, \|\boldsymbol{v}_h\|_{0,K}, \tag{9}$$

$$\|[\![p_h]\!]\|_{0,F} \leq C\, h_F^{-1/2} \|p_h\|_{0,\omega_F}, \tag{10}$$

$$h_K\, |\boldsymbol{v}_h|_{1,K} \leq C\, \|\boldsymbol{v}_h\|_{0,K}. \tag{11}$$

Proof. Results (9) and (11) follow from Lemma 1.138 in [16], and (10) follows from the mesh regularity and Lemma 1. □

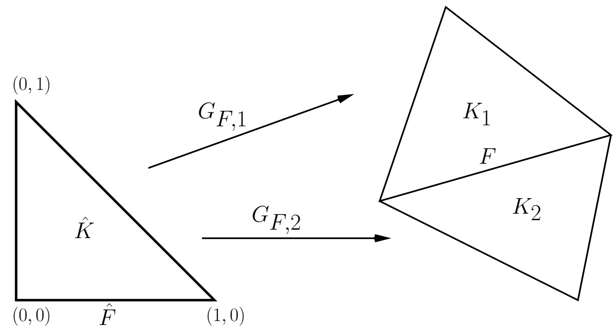

FIGURE 1. Affine transformation $G_{F,i}$, $i = 1, 2$.

Lemma 3.

$$\|v - \Pi_K v\|_{0,K} \leq Ch_K |v|_{1,K} \qquad \forall v \in H^1(K), \tag{12}$$

$$\|\Pi_K v\|_{0,K} \leq C\|v\|_{0,K} \qquad \forall v \in L^2(K), \tag{13}$$

$$\|\Pi_K v\|_{\infty,K} \leq Ch_K^{-1}\|v\|_{0,K} \qquad \forall v \in L^2(K). \tag{14}$$

Proof. Estimates (12) and (13) follow from Lemma 1.131 and Proposition 1.134 in [16], respectively. Estimate (14) is a consequence of mesh regularity, Cauchy-Schwarz's inequality and the definition of Π_K. □

Now, we define functions with support on an triangle or on an edge which will be used to prove the local efficiency of the a posteriori error estimator. Given $K \in \mathcal{T}_h$, we introduce the elementary bubble function, b_K, by

$$b_K := 27 \prod_{x \in \mathcal{N}(K)} \lambda_x,$$

where λ_x denotes the barycentric coordinates associated to the vertex x. To define and edge bubble function, we denote by $\hat{K}$ the unitary reference triangle element and we set

$$b_{\hat{F}} := 4\,\hat{\lambda}_3\hat{\lambda}_1 \quad \text{on} \quad \hat{K}\,,$$

where $\hat{F} := \{(t,0) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$. Next, given $F \in \mathcal{E}_h$ and assuming that $\omega_F = K_1 \cup K_2$, let $G_{F,i}$ be the (orientation preserving) affine transformation (see Figure 1) such that $G_{F,i}(\hat{K}) = K_i$ and $G_{F,i}(\hat{F}) = F$, $i = 1, 2$. Thus the bubble function associated to an edge F reads

$$b_F := \begin{cases} b_{\hat{F}} \circ G_{F,i}^{-1} & \text{on} \quad K_i,\ i = 1, 2, \\ 0 & \text{on} \quad \Omega \setminus \omega_F. \end{cases}$$

Let $\hat{\Pi} := \{(x,0) : x \in \mathbb{R}\}$ and $\hat{Q} : \mathbb{R}^2 \to \hat{\Pi}$ be the orthogonal projection. We introduce a lifting operator acting on functions defined on the reference element as follows $\hat{P}_{\hat{F}} : \mathbb{P}_k(\hat{F}) \to \mathbb{P}_k(\hat{K})$

$$\hat{s} \longmapsto \hat{P}_{\hat{F}}(\hat{s}) = \hat{s} \circ \hat{Q}.$$

Next, we propose the lifting operator on the real element $K_i \subseteq \omega_F$, $P_{F,K_i} : \mathbb{P}_k(F) \to \mathbb{P}_k(K_i)$, given by

$$P_{F,K_i}(s) = \hat{P}_{\hat{F}}(s \circ G_{F,i}) \circ G_{F,i}^{-1},$$

from which we define $P_F : \mathbb{P}_k(F) \to \mathbb{P}_k(\omega_F)$ by

$$s \in \mathbb{P}_k(F) \longmapsto P_F(s) := \begin{cases} P_{F,K_1}(s) & \text{in } K_1, \\ P_{F,K_2}(s) & \text{in } K_2. \end{cases}$$

If $\boldsymbol{s} := (s_1, s_2) \in \mathbb{P}_k(F)^2$, then we define $\mathcal{P}_F : \mathbb{P}_k^2(F) \to \mathbb{P}_k^2(\omega_F)$ by

$$\mathcal{P}_F(\boldsymbol{s}) = (P_F(s_1), P_F(s_2)) .$$

From the previous definitions and using standard scaling arguments, the following equivalences hold.

Lemma 4. *Let $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$. Given $v \in \mathbb{P}_k(K)$ and $s \in \mathbb{P}_l(F)$ with $k, l \geq 0$, the following estimates hold*

$$C \, \|v\|_{0,K} \leq \sup_{\substack{w \in \mathbb{P}_k(K) \\ w \neq 0}} \frac{(v, b_K w)}{\|w\|_{0,K}} \leq \|v\|_{0,K}, \tag{15}$$

$$C \, \|s\|_{0,F} \leq \sup_{\substack{r \in \mathbb{P}_k(F) \\ r \neq 0}} \frac{(s, b_F r)}{\|r\|_{0,F}} \leq \|s\|_{0,F}, \tag{16}$$

$$C h_K^{-1} \, \|b_K v\|_{0,K} \leq |b_K v|_{1,K} \leq C h_K^{-1} \, \|b_K v\|_{0,K}, \tag{17}$$

$$C h_K^{-1} \, \|b_F P_F(s)\|_{0,K} \leq |b_F P_F(s)|_{1,K} \leq C h_K^{-1} \, \|b_F P_F(s)\|_{0,K}, \tag{18}$$

$$\|b_F P_F(s)\|_{0,K} \leq C \, h_K^{1/2} \|s\|_{0,F}. \tag{19}$$

Proof. See Lemma 5.1 in [32]. □

3. The residual local projection method

Now we introduce a stabilized finite element method for (1). Such scheme is a variant of the RELP method introduced in [4], in which the difference lays on the redefinition of the boundary stabilization terms. The RELP method in this work reads: *Find $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that*

$$\mathbf{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \mathbf{F}(\boldsymbol{v}_h, q_h), \tag{20}$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where the form $\mathbf{B}(\cdot, \cdot)$ is given by

$$\begin{aligned} \mathbf{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) := \; & \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) - (q_h, \nabla \cdot \boldsymbol{u}_h) \\ & - \sum_{K \in \mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x} \cdot (\nabla \boldsymbol{u}_h) \Pi_K \boldsymbol{u}_h + \boldsymbol{x} \cdot \nabla p_h), \chi_h(-\boldsymbol{x} \cdot (\nabla \boldsymbol{v}_h) \Pi_K \boldsymbol{u}_h + q_h) \right)_K \\ & + \sum_{K \in \mathcal{T}_h} \frac{\gamma_K}{\nu} \left(\chi_h(\boldsymbol{x} \, \nabla \cdot \boldsymbol{u}_h), \chi_h(\boldsymbol{x} \, \nabla \cdot \boldsymbol{v}_h) \right)_K \\ & - \sum_{F \in \mathcal{E}_h} \tau_F \left([\![-\nu \, \partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \, \boldsymbol{n}]\!], [\![\nu \, \partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \, \boldsymbol{n}]\!] \right)_F , \end{aligned}$$

and $\mathbf{F}(\cdot)$ by

$$\mathbf{F}(\boldsymbol{v}_h, q_h) := (\boldsymbol{f}, \boldsymbol{v}_h) - \sum_{K \in \mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x} \cdot \Pi_K \boldsymbol{f}), \chi_h(-\boldsymbol{x} \cdot (\nabla \boldsymbol{v}_h) \Pi_K \boldsymbol{u}_h + q_h) \right)_K . \tag{21}$$

Here χ_h represents the *fluctuation* operator defined by $\chi_h := \mathrm{I} - \Pi_K$, where I is the identity operator, and the element–wise stabilization parameters α_K and γ_K are given by

$$\alpha_K := \frac{1}{\max\{1, Pe_K\}} \qquad \text{and} \qquad \gamma_K := \frac{1}{\max\left\{1, \dfrac{Pe_K}{24}\right\}},$$

where

$$Pe_K := \frac{|\boldsymbol{u}_h|_K h_K}{18\,\nu} \quad \text{with} \quad |\boldsymbol{u}_h|_K := \frac{\|\boldsymbol{u}_h\|_{0,K}}{|K|^{\frac{1}{2}}}.$$

Also, the edge–wise parameter τ_F is defined by

$$\tau_F := \begin{cases} \dfrac{h_F}{12\,\nu} & \text{if } |\boldsymbol{u}_h|_F = 0\,, \\[2ex] \dfrac{1}{2\,|\boldsymbol{u}_h|_F} - \dfrac{1}{|\boldsymbol{u}_h|_F\,(1-\exp(Pe_F))}\Big(1 + \dfrac{1}{Pe_F}(1-\exp(Pe_F)\Big) & \text{otherwise}\,. \end{cases}$$

Here

$$Pe_F := \frac{|\boldsymbol{u}_h|_F h_F}{\nu} \quad \text{with} \quad |\boldsymbol{u}_h|_F := \frac{\|\boldsymbol{u}_h\|_{0,F}}{h_F^{1/2}}.$$

We note that τ_F satisfies (see Lemma 2 in [11])

$$\tau_F \le C\,\frac{h_F}{\nu}, \tag{22}$$

for all $F \in \mathcal{E}_h$ with a positive constant C which is independent of h and ν.

Mimicking what was done in the continuos case, method (20) is reformulate using the operator $F_h : \mathbf{V}_h \times Q_h \longrightarrow (\mathbf{V}_h \times Q_h)'$ which is defined by

$$\begin{aligned} &\langle F_h(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)\rangle := \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)\rangle \\ &-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\Big[\alpha_K\,(\chi_h(-\nu\,\Delta\boldsymbol{u}_h\cdot\boldsymbol{x} + (\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h\cdot\boldsymbol{x} + \nabla p_h\cdot\boldsymbol{x} - \Pi_K\boldsymbol{f}\cdot\boldsymbol{x}), \chi_h(\nabla q_h\cdot\boldsymbol{x} - (\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h\cdot\boldsymbol{x}))_K \\ &+\gamma_K\,(\chi_h(\nabla\cdot\boldsymbol{u}_h\,\boldsymbol{x}), \chi_h(\nabla\cdot\boldsymbol{v}_h\,\boldsymbol{x}))_K\Big] - \sum_{F\in\mathcal{E}_h}\tau_F\,([\![-\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h + p_h\boldsymbol{n}]\!],[\![\nu\,\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\boldsymbol{n}]\!])_F. \end{aligned}$$

As a result, (20) can be rewrite as follows: *Find* $(\boldsymbol{u}_h,p_h) \in \mathbf{V}_h \times Q_h$ *such that*

$$\langle F_h(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)\rangle = 0 \qquad \forall(\boldsymbol{v}_h,q_h) \in \mathbf{V}_h \times Q_h.$$

Before heading to the proof of the existence and the uniqueness of a solution for RELP method (20), we need some auxiliary results. We define the operator $\mathcal{P} : \mathbf{V}_h \longrightarrow Q_h$ by

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\big(\chi_h(\mathcal{P}(\boldsymbol{u}_h)),\chi_h(q_h)\big)_K + \sum_{F\in\mathcal{E}_h}\tau_F\,\big([\![\mathcal{P}(\boldsymbol{u}_h)]\!],[\![q_h]\!]\big)_F \\ = \quad & -(q_h,\nabla\cdot\boldsymbol{u}_h) - \sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\,(\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - \Pi_K\boldsymbol{f})),\chi_h(q_h))_K \\ & - \sum_{F\in\mathcal{E}_h}\tau_F\,\,([\![-\partial_{\boldsymbol{n}}\boldsymbol{u}_h]\!],[\![q_h\,\boldsymbol{n}]\!])_F\ , \end{aligned} \tag{23}$$

for all $\boldsymbol{u}_h \in \mathbf{V}_h$, $q_h \in Q_h$. Observe that the $\mathcal{P}$ is well-defined from Lax–Milgram's Theorem with the norm

$$\|q_h\|_* := \left\{\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\|\chi_h(q_h)\|^2_{0,K} + \sum_{F\in\mathcal{E}_h}\tau_F\,\|[\![q_h]\!]\|^2_{0,F}\right\}^{1/2}.$$

Also, define the mapping $\mathcal{N} : \mathbf{V}_h \longrightarrow \mathbf{V}_h$ by

$$\begin{aligned}(\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{v}_h) &= \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (\mathcal{P}(\boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h) - (\boldsymbol{f}, \boldsymbol{v}_h) \\ &+ \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu} \left(\chi_h(\boldsymbol{x}\,\nabla \cdot \boldsymbol{u}_h), \chi_h(\boldsymbol{x}\,\nabla \cdot \boldsymbol{v}_h)\right)_K \\ &- \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x} \cdot ((\nabla \boldsymbol{u}_h)\Pi_k \boldsymbol{u}_h - \Pi_K \boldsymbol{f}) + \mathcal{P}(\boldsymbol{u}_h)), \chi_h(-\boldsymbol{x} \cdot ((\nabla \boldsymbol{v}_h)\Pi_K \boldsymbol{u}_h))\right)_K \\ &- \sum_{F\in\mathcal{E}_h} \tau_F \left([\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!], [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h]\!]\right)_F \,,\end{aligned} \tag{24}$$

for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h$.

The next result provides a characterization of the solution of (20) with respect to the operators $\mathcal{P}$ and $\mathcal{N}$.

Lemma 5. *The pair $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is a solution of problem (20) if and only if $\mathcal{N}(\boldsymbol{u}_h) = \mathbf{0}$ and $p_h = \mathcal{P}(\boldsymbol{u}_h)$.*

Proof. Let $\boldsymbol{u}_h \in \mathbf{V}_h$ such that $\mathcal{N}(\boldsymbol{u}_h) = \mathbf{0}$ and let $p_h = \mathcal{P}(\boldsymbol{u}_h)$. Adding (23) and (24) we see that $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is a solution of problem (20). Now assume $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is a solution of (20). Then, taking $\boldsymbol{v}_h = \mathbf{0}$ in (20) it holds

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(p_h), \chi_h(q_h)\right)_K + \sum_{F\in\mathcal{E}_h} \tau_F \left([\![p_h]\!], [\![q_h]\!]\right)_F = \\ &-(q_h, \nabla \cdot \boldsymbol{u}_h) - \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x} \cdot ((\nabla \boldsymbol{u}_h)\Pi_k \boldsymbol{u}_h - \Pi_K \boldsymbol{f})), \chi_h(q_h)\right)_K \\ &- \sum_{F\in\mathcal{E}_h} \tau_F \left([\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h]\!], [\![q_h \boldsymbol{n}]\!]\right)_F\end{aligned}$$

and hence, since $\mathcal{P}$ is well defined, $p_h = \mathcal{P}(\boldsymbol{u}_h)$. Next, taking $q_h = 0$ in (20), we arrive at $\mathcal{N}(\boldsymbol{u}_h) = \mathbf{0}$ and the result follows. □

We are now ready to prove the well-posedness of (20). The proof follows closely the arguments presented in [4].

Theorem 6. *There is a positive constant $\tilde{C}$, which is independent of h and ν, such that problem (20) admits at least one solution $(\boldsymbol{u}_h, p_h)$ provided*

$$\frac{h^{1-\kappa}}{\nu^{3/2}} \left\{ \frac{1}{\nu}\|\boldsymbol{f}\|^2_{-1,\Omega} + \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x} \cdot \Pi_K \boldsymbol{f})\|^2_{0,K} \right\}^{1/2} \leq \tilde{C}\,, \tag{25}$$

where $0 < \kappa < 1$. Moreover, the solution of problem (20) is unique provided that ν is large enough.

Proof. Let $R > 0$ and $\boldsymbol{u}_h \in \mathbf{V}_h$, with $|\boldsymbol{u}_h|_{1,\Omega} = R$, be arbitrary chosen and denote

$$
\begin{aligned}
x &:= \left\{ \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h) + \mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \right\}^{1/2}, \\
y &:= \left\{ \sum_{F\in\mathcal{E}_h} \tau_F \, \|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \right\}^{1/2}, \\
z &:= \left\{ \frac{1}{\nu}\|\boldsymbol{f}\|_{-1,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f})\|_{0,K}^2 \right\}^{1/2}, \\
w &:= \left\{ \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h)\|_{0,K}^2 \right\}^{1/2}.
\end{aligned}
$$

Taking $q_h = \mathcal{P}(\boldsymbol{u}_h)$ in (23) gives

$$
\begin{aligned}
-(\mathcal{P}(\boldsymbol{u}_h), \nabla\cdot\boldsymbol{u}_h) &= \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - \Pi_K\boldsymbol{f}) + \mathcal{P}(\boldsymbol{u}_h)), \chi_h(\mathcal{P}(\boldsymbol{u}_h))\right)_K \\
&\quad + \sum_{F\in\mathcal{E}_h} \tau_F ([\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!], [\![\mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!])_F.
\end{aligned}
$$

From Cauchy-Schwarz's inequality, (24) and the identity $((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) = -\dfrac{1}{2}(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h)$, we get

$$
\begin{aligned}
(\mathcal{N}(\boldsymbol{u}_h), \boldsymbol{u}_h) &= \nu|\boldsymbol{u}_h|_{1,\Omega}^2 + ((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - (\boldsymbol{f}, \boldsymbol{u}_h) \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - \Pi_K\boldsymbol{f}) + \mathcal{P}(\boldsymbol{u}_h)), \chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h) + \mathcal{P}(\boldsymbol{u}_h))\right)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tau_F \, \|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\
&\geq \nu|\boldsymbol{u}_h|_{1,\Omega}^2 + ((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) - \frac{1}{\sqrt{\nu}}\|\boldsymbol{f}\|_{-1,\Omega}\sqrt{\nu}|\boldsymbol{u}_h|_{1,\Omega} \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h) + \mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \\
&\quad - \sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f}), \chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h) + \mathcal{P}(\boldsymbol{u}_h))\right)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tau_F \, \|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\
&\geq \frac{\nu}{2}|\boldsymbol{u}_h|_{1,\Omega}^2 + ((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{u}_h) + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h) + \mathcal{P}(\boldsymbol{u}_h))\|_{0,K}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\gamma_K}{\nu} \|\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h} \tau_F \, \|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + \mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2 \\
&\quad - \frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{\nu} \|\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f})\|_{0,K}^2 - \frac{1}{2\nu}\|\boldsymbol{f}\|_{-1,\Omega}^2 \\
&\geq \frac{\nu}{2}R^2 + \frac{1}{2}x^2 + y^2 + w^2 - \frac{1}{2}z^2 - \frac{1}{2}(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h). \qquad (26)
\end{aligned}
$$

Now, if we take $(\boldsymbol{v}_h, q_h) = (\boldsymbol{0}, \mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))$ in (20), use Cauchy–Schwarz's inequality, Lemma 3, (23), the fact that $\alpha_K \leq 1$, and (6) with the mesh regularity assumption, we get

$$
\begin{aligned}
& |(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)| \le |(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h-\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))|+|(\nabla\cdot\boldsymbol{u}_h,\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))| \\
\le\;& \sqrt{2}|\boldsymbol{u}_h|_{1,\Omega}\,\|\boldsymbol{u}_h\cdot\boldsymbol{u}_h-\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\|_{0,\Omega} \\
& +\left|\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}(\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h-\Pi_K\boldsymbol{f})+\mathcal{P}(\boldsymbol{u}_h)),\chi_h(\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)))_K\right| \\
& +\left|\sum_{F\in\mathcal{E}_h}\tau_F\,\|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h+\mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\,\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\right| \\
\le\;& C\sqrt{\nu}R\left\{\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2} \\
& +\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\|\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h-\Pi_K\boldsymbol{f})+\mathcal{P}(\boldsymbol{u}_h))\|_{0,K}\,\|\chi_h(\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h))\|_{0,K} \\
& +\sum_{F\in\mathcal{E}_h}\tau_F\,\|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h+\mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\,\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F} \\
\le\;& C\sqrt{\nu}R\left\{\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2}+C\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\|\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f})\|_{0,K}\,h_K|\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|_{1,K} \\
& +C\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}\|\chi_h(\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h)+\mathcal{P}(\boldsymbol{u}_h))\|_{0,K}\,h_K|\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|_{1,K} \\
& +\sum_{F\in\mathcal{E}_h}\tau_F\,\|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h+\mathcal{P}(\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}\,\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F} \\
\le\;& C\sqrt{\nu}R\left\{\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2}+Cz\left\{\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\omega_K}^2\right\}^{1/2} \\
& +Cx\left\{\sum_{K\in\mathcal{T}_h}\frac{\alpha_K}{\nu}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,\omega_K}^2\right\}^{1/2}+y\left\{\sum_{F\in\mathcal{E}_h}\tau_F\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\boldsymbol{n}]\!]\|_{0,F}^2\right\}^{1/2} \\
\le\;& C\{\sqrt{\nu}R+x+y+z\}\left\{\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2+\sum_{F\in\mathcal{E}_h}\tau_F\|[\![\mathcal{J}_h(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)-\boldsymbol{u}_h\cdot\boldsymbol{u}_h]\!]\|_{0,F}^2\right\}^{1/2}.
\end{aligned}
$$

As a result of the above estimate, (22), (7) and the mesh regularity, we arrive at

$$
|(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)|\le\frac{C}{\sqrt{\nu}}\{\sqrt{\nu}R+x+y+z\}\left\{\sum_{K\in\mathcal{T}_h}h_K^2|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K}^2\right\}^{1/2}. \tag{27}
$$

Moreover, using the local inverse inequality $\|v_h\|_{\infty,K}\le Ch_K^{-\frac{2}{q}}\|v_h\|_{0,q,K}$ for all $1\le q\le\infty$ (see [16]) and the Sobolev embedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$ for all $2\le q\le\infty$, we obtain

$$
\begin{aligned}
|\boldsymbol{u}_h\cdot\boldsymbol{u}_h|_{1,K} &= \|\nabla(\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\|_{0,K}=2\|\nabla(\boldsymbol{u}_h)\boldsymbol{u}_h\|_{0,K} \\
&\le C|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{\infty,K}\le Ch_K^{-\frac{2}{q}}|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{q,K} \\
&\le Ch_K^{-\frac{2}{q}}|\boldsymbol{u}_h|_{1,K}\,\|\boldsymbol{u}_h\|_{q,\Omega}\le Ch_K^{-\frac{2}{q}}|\boldsymbol{u}_h|_{1,K}\,|\boldsymbol{u}_h|_{1,\Omega}\,,
\end{aligned}
$$

and then from (26) and (27), it holds

$$
\begin{aligned}
(\mathcal{N}(\boldsymbol{u}_h),\boldsymbol{u}_h) &\geq \frac{\nu}{2}R^2+\frac{1}{2}x^2+w^2+y^2-\frac{1}{2}z^2-\frac{1}{2}(\nabla\cdot\boldsymbol{u}_h,\boldsymbol{u}_h\cdot\boldsymbol{u}_h)\\
&\geq \frac{\nu}{2}R^2+\frac{1}{2}x^2+w^2+y^2-\frac{1}{2}z^2\\
&\quad -\frac{C}{2\sqrt{\nu}}\{\sqrt{\nu}R+x+y+z\}\left\{\sum_{K\in\mathcal{T}_h}h_K^{2-2\kappa}|\boldsymbol{u}_h|_{1,K}^2\right\}^{\frac{1}{2}}|\boldsymbol{u}_h|_{1,\Omega}\\
&\geq \frac{\nu}{2}R^2+\frac{1}{2}x^2+w^2+y^2-\frac{1}{2}z^2-\frac{C}{\sqrt{\nu}}h^{1-\kappa}\{\sqrt{\nu}R+x+y+z\}\,R^2\\
&\geq \frac{\nu}{2}R^2+\frac{1}{2}x^2+w^2+y^2-\frac{1}{2}z^2-Ch^{1-\kappa}R^3-\frac{C}{\sqrt{\nu}}h^{1-\kappa}\{x+y+z\}\,R^2\\
&\geq \frac{\nu}{2}R^2+\frac{1}{2}x^2+w^2+y^2\\
&\quad -\frac{1}{2}z^2-Ch^{1-\kappa}R^3-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}z^2-\frac{3}{2}\frac{C^2}{\nu}h^{2(1-\kappa)}R^4\\
&\geq \frac{\nu}{2}R^2+w^2+\frac{1}{2}y^2-z^2-Ch^{1-\kappa}R^3-\frac{3}{2}\frac{C^2}{\nu}h^{2(1-\kappa)}R^4,
\end{aligned}
$$

where $\kappa:=\frac{2}{q}$. Now, set $R:=\dfrac{\nu}{MCh^{1-\kappa}}$ for an integer $M\geq 6$, and observe that M satisfies

$$
\frac{1}{2}-\frac{1}{M}-\frac{3}{2M^2}\geq\frac{1}{4}.
$$

By selecting $\tilde{C}=\frac{1}{2MC}$ in (25) and observing that assumption (25) leads to

$$
\frac{2MCh^{1-\kappa}}{\nu^{3/2}}z\leq 1\,,
$$

we conclude (using the definition of R above) that $z\leq\dfrac{\sqrt{\nu}}{2}R$. Gathering previous inequalities together, it holds

$$
\begin{aligned}
(\mathcal{N}(\boldsymbol{u}_h),\boldsymbol{u}_h) &\geq (\frac{1}{2}-\frac{1}{M}-\frac{3}{2M^2})\nu R^2-z^2+\frac{1}{2}y^2+w^2\\
&\geq \frac{\nu}{4}R^2-z^2+\frac{1}{2}y^2+w^2\geq\frac{1}{2}y^2+w^2\geq 0\,.
\end{aligned}
$$

Thus Brouwer's fixed point Theorem implies the existence of $\boldsymbol{u}_h\in\mathbf{V}_h$ with $|\boldsymbol{u}_h|_{1,\Omega}\leq R$ and $\mathcal{N}(\boldsymbol{u}_h)=\mathbf{0}$. The uniqueness of solution follows from Banach's fixed point Theorem using the arguments presented in [20] (see Appendix A for a proof). □

4. A residual error estimator

In this section, we propose a residual a posteriori error estimator for the method (20). The analysis follows mainly the ideas introduced by Verfürth in [32]. For sake of simplicity, we assume that

(F) $\boldsymbol{f}$ is a piecewise polynomial function, i.e., $\boldsymbol{f}\Big|_K\in\mathbb{P}_l(K)^2$, $l\in\mathbb{N}\cup\{0\}$, $\forall K\in\mathcal{T}_h$.

It worth mentioning that such an assumption may be relaxed. Indeed, if we only assume $\boldsymbol{f}\in L^2(\Omega)^2$, for instance, estimates (31), (32) (see Theorem 8 below) will include a correction term of type $h_K\,\|\boldsymbol{f}-\Pi_K\boldsymbol{f}\|_{0,K}$, for $K\in\mathcal{T}_h$, which is in general a high order term.

To introduce the error estimator, we define for each $K \in \mathcal{T}_h$ and each $F \in \mathcal{E}_h$, the following residual quantities

$$\mathcal{R}_K := \Big(\boldsymbol{f} + \nu\Delta \boldsymbol{u}_h - (\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - \nabla p_h\Big)\Big|_K \quad \text{and} \quad \mathcal{R}_F := [\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + p_h\boldsymbol{n}]\!]_F \,.$$

Using these definitions, the residual–based error estimator reads

$$\eta := \left\{ \sum_{K\in\mathcal{T}_h} \eta_K^2 \right\}^{\frac{1}{2}}, \tag{28}$$

where

$$\eta_K^2 := \frac{h_K^2}{\nu}\|\mathcal{R}_K\|_{0,K}^2 + \nu\,\|\nabla\cdot \boldsymbol{u}_h\|_{0,K}^2 + \frac{1}{2}\sum_{F\in\mathcal{E}(K)\cap\mathcal{E}_h}\frac{h_F}{\nu}\,\|\mathcal{R}_F\|_{0,F}^2.$$

The next result set the framework within the analysis of the estimator is established. Such a result is due to Verfürth (see Proposition 2.1 in [32]).

Theorem 7. *Let $(\boldsymbol{u},p) \in \mathbf{V}\times Q$ be a non–singular solution of equation (2). Set*

$$R := \min\Big\{R_0, \gamma^{-1}\|D_{\boldsymbol{u},p}F(\boldsymbol{u},p)^{-1}\|^{-1}_{\mathcal{L}((\mathbf{V}\times Q)',(\mathbf{V}\times Q))}, 2\gamma^{-1}\|D_{\boldsymbol{u},p}F(\boldsymbol{u},p)\|_{\mathcal{L}((\mathbf{V}\times Q),(\mathbf{V}\times Q)')}\Big\}\,.$$

Then, the following error estimates hold for all $(\boldsymbol{v}_h,q_h)\in\mathbb{B}((\boldsymbol{u},p),R)$

$$\|(\boldsymbol{u}-\boldsymbol{v}_h, p-q_h)\| \;\leq\; 2\,\|D_{\boldsymbol{u},p}F(\boldsymbol{u},p)^{-1}\|_{\mathcal{L}((\mathbf{V}\times Q)',(\mathbf{V}\times Q))}\,\|F(\boldsymbol{v}_h,q_h)\|_{(\mathbf{V}\times Q)'}, \tag{29}$$

$$\|(\boldsymbol{u}-\boldsymbol{v}_h, p-q_h)\| \;\geq\; \frac{1}{2}\,\|D_{\boldsymbol{u},p}F(\boldsymbol{u},p)\|^{-1}_{\mathcal{L}((\mathbf{V}\times Q),(\mathbf{V}\times Q)')}\,\|F(\boldsymbol{v}_h,q_h)\|_{(\mathbf{V}\times Q)'}. \tag{30}$$

We are ready to present the main result of this section.

Theorem 8. *Let $(\boldsymbol{u},p)$ be a regular solution of (2) and $(\boldsymbol{u}_h,p_h)$ be the solution of (20). If we assume that $(\boldsymbol{u}_h,p_h)\in\mathbb{B}((\boldsymbol{u},p),R)$, for R sufficiently small, then the following a posteriori error estimates hold*

$$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\| \;\leq\; C_1\,\max\left\{1, \frac{\|\boldsymbol{u}_h\|_{0,\Omega}}{\nu}\right\}\eta_H, \tag{31}$$

$$\eta \;\leq\; C_2\,\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|, \tag{32}$$

where

$$\eta_H^2 := \sum_{K\in\mathcal{T}_h}\left[\eta_K^2 + \frac{h_K^2}{\nu}\left(\|(\nabla\boldsymbol{u}_h)\chi_h(\boldsymbol{u}_h)\|_{0,K}^2 + \frac{h_K^2}{\nu^2}\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,K}^2\right)\right],$$

and η is defined in (28). The positive constants C_1 and C_2 are independent of h and ν, but they may depend on $\boldsymbol{u}$ and p.

Proof. Lower bound: Define the finite–dimensional subspace $\mathcal{B}_h \subset \mathbf{V}\times Q$ as follows

$$\mathcal{B}_h := \operatorname{span}\big\{(b_K\boldsymbol{v},0), (b_F\mathcal{P}_F(\boldsymbol{s}),0), (0,b_K r) : \boldsymbol{v}\in\mathbb{P}_1(K)^2,\, \boldsymbol{s}\in\mathbb{P}_l(F)^2,\, r\in\mathbb{P}_0(K),\, \forall K\in\mathcal{T}_h,\, \forall F\in\mathcal{E}_h\big\},$$

with $l=0,1$. From Lemma 4, we get

$$
\begin{aligned}
\sqrt{\nu}\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,K} &\leq C \sup_{r\in\mathbb{P}_0(K)\setminus\{0\}} \frac{(\nabla\cdot\boldsymbol{u}_h, b_K r)_K}{\frac{1}{\sqrt{\nu}}\,\|r\|_{0,K}} \\
&\leq C \sup_{r\in\mathbb{P}_0(K)\setminus\{0\}} \frac{\langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{0},b_K r)\rangle}{\frac{1}{\sqrt{\nu}}\,\|r\|_{0,K}} \\
&\leq C \sup_{\substack{(\boldsymbol{v},q)\in\mathcal{B}_h|_K\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle, \qquad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sqrt{\nu}}\,h_K\,\|R_K\|_{0,K} &\leq C \sup_{\boldsymbol{w}\in\mathbb{P}_1(K)^2\setminus\{0\}} \frac{(R_K, b_K\boldsymbol{w})_K}{\sqrt{\nu}\,|b_K\boldsymbol{w}|_{1,K}} \\
&\leq C \sup_{\boldsymbol{w}\in\mathbb{P}_1(K)^2\setminus\{0\}} \frac{\langle F(\boldsymbol{u}_h,p_h),(b_K\boldsymbol{w},\boldsymbol{0})\rangle}{\sqrt{\nu}\,|b_K\boldsymbol{w}|_{1,K}} \\
&\leq C \sup_{\substack{(\boldsymbol{v},q)\in\mathcal{B}_h|_K\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle. \qquad (34)
\end{aligned}
$$

In addition, using estimates (15)–(19) and (34), it holds

$$
\begin{aligned}
&\frac{1}{\sqrt{\nu}}\,h_F^{1/2}\,\|R_F\|_{0,F} \\
\leq\;& C\,h_F^{1/2} \sup_{\boldsymbol{s}\in\mathbb{P}_l(F)^2\setminus\{0\}} \frac{(R_F, b_F s)_F}{\sqrt{\nu}\,\|\boldsymbol{s}\|_{0,F}} \\
\leq\;& C\,h_F \sup_{\boldsymbol{s}\in\mathbb{P}_l(F)^2\setminus\{0\}} \frac{\left\{\langle F(\boldsymbol{u}_h,p_h),(b_F\mathcal{P}_F(\boldsymbol{s}),\boldsymbol{0})\rangle - \sum_{K\in\omega_F}(R_K, b_F\mathcal{P}_F(\boldsymbol{s}))_K\right\}}{\sqrt{\nu}\,\|b_F\mathcal{P}_F(\boldsymbol{s})\|_{0,\omega_F}} \\
\leq\;& C \sup_{\boldsymbol{s}\in\mathbb{P}_l(F)^2\setminus\{0\}} \frac{\left\{\langle F(\boldsymbol{u}_h,p_h),(b_F\mathcal{P}_F(\boldsymbol{s}),0)\rangle - \sum_{K\in\omega_F}(R_K, b_F\mathcal{P}_F(\boldsymbol{s}))_K\right\}}{\sqrt{\nu}\,|b_F\mathcal{P}_F(\boldsymbol{s})|_{1,\omega_F}} \\
\leq\;& C \sup_{\substack{(\boldsymbol{v},q)\in\mathcal{B}_h|_K\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle + \frac{h_F}{\sqrt{\nu}} \sum_{K\in\omega_F} \|R_K\|_{0,K} \\
\leq\;& C \sup_{\substack{(\boldsymbol{v},q)\in\mathcal{B}_h|_{\omega_F}\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle. \qquad (35)
\end{aligned}
$$

Observe that inequalities (33)–(35) imply

$$
\eta_K \leq C \sup_{\substack{(\boldsymbol{v},q)\in\mathcal{B}_h|_{\omega_K}\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle. \qquad (36)
$$

Finally, as $\sum_{K\in\mathcal{T}_h} \eta_K^2 \leq \left[\sum_{K\in\mathcal{T}_h} \eta_K\right]^2$, we obtain from (36) that

$$
\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2} \leq \sum_{K\in\mathcal{T}_h} \eta_K \leq C\,\|F(\boldsymbol{u}_h,p_h)\|_{\mathcal{B}_h'}, \qquad (37)
$$

and from (30) the result follows.

Upper bound: From (3)–(5), we get

$$
\begin{aligned}
&\sup_{\substack{(\boldsymbol{v},q)\in\mathbf{V}\times Q\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v},q)\rangle \\
=\;& \sup_{\substack{(\boldsymbol{v},q)\in\mathbf{V}\times Q\\ \|(\boldsymbol{v},q)\|=1}} \left\{ \sum_{K\in\mathcal{T}_h} \Big[(-\nu\,\Delta\boldsymbol{u}_h+(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h+\nabla p_h-\boldsymbol{f},\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v})_K-(\nabla\cdot\boldsymbol{u}_h,q)_K\Big] \right.\\
&\left. +\sum_{F\in\mathcal{E}_h}([\![-\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h+p_h\boldsymbol{n}]\!],\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v})_F \right\}\\
\leq\;& C\left\{\sum_{K\in\mathcal{T}_h}\eta_K^2\right\}^{1/2}. \qquad (38)
\end{aligned}
$$

Next, using (37) and (38) we get

$$
\sup_{\substack{(\boldsymbol{v},q)\in\mathbf{V}\times Q\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v},q)\rangle \leq C\,\|F(\boldsymbol{u}_h,p_h)\|_{\mathcal{B}_h'}, \qquad (39)
$$

and by considering an arbitrary element $(\boldsymbol{v},q)\in\mathbf{V}\times Q$ with $\|(\boldsymbol{v},q)\|=1$, we arrive at

$$
\begin{aligned}
&\langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v},q)\rangle\\
=\;&\langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v},q)\rangle+\langle F(\boldsymbol{u}_h,p_h)-F_h(\boldsymbol{u}_h,p_h),(\mathcal{I}_h\boldsymbol{v},0)\rangle\\
\leq\;&\sup_{\substack{(\boldsymbol{v},q)\in\mathbf{V}\times Q\\ \|(\boldsymbol{v},q)\|=1}} \langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v},q)\rangle\\
&+\|\mathcal{I}_h\|_{\mathcal{L}(\mathbf{V},\mathbf{V}_h)}\|F(\boldsymbol{u}_h,p_h)-F_h(\boldsymbol{u}_h,p_h)\|_{(\mathbf{V}_h\times Q_h)'}.
\end{aligned}
$$

Thereby, from (39), we get

$$
\begin{aligned}
&\|F(\boldsymbol{u}_h,p_h)\|_{(\mathbf{V}\times Q)'}\\
\leq\;& C\,\|F(\boldsymbol{u}_h,p_h)\|_{\mathcal{B}_h'}+\|\mathcal{I}_h\|_{\mathcal{L}(\mathbf{V},\mathbf{V}_h)}\,\|F(\boldsymbol{u}_h,p_h)-F_h(\boldsymbol{u}_h,p_h)\|_{(\mathbf{V}_h\times Q_h)'}. \qquad (40)
\end{aligned}
$$

Now, given $(\boldsymbol{v}_h,q_h)\in\mathcal{B}_h$ and integrating by parts, we obtain that

$$
\begin{aligned}
&\langle F(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)\rangle\\
=\;&\sum_{K\in\mathcal{T}_h}\Big[(-\nu\,\Delta\boldsymbol{u}_h+(\nabla\boldsymbol{u}_h)\boldsymbol{u}_h+\nabla p_h-\boldsymbol{f},\boldsymbol{v}_h)_K-(\nabla\cdot\boldsymbol{u}_h,q_h)_K\Big]\\
&+\sum_{F\in\mathcal{E}_h}([\![-\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_h+p_h\boldsymbol{n}]\!],\boldsymbol{v}_h)_F\\
=\;&-\sum_{K\in\mathcal{T}_h}\Big[(\mathcal{R}_K,\boldsymbol{v}_h)_K+(\nabla\cdot\boldsymbol{u}_h,q_h)_K\Big]+\sum_{F\in\mathcal{E}_h}(\mathcal{R}_F,\boldsymbol{v}_h)_F. \qquad (41)
\end{aligned}
$$

Next, we will estimate the terms on the right-hand side of (41). To this end, we use estimates (15)–(19) to get

$$
\begin{aligned}
(\mathcal{R}_K,b_K\boldsymbol{w})_K &\leq h_K\|\mathcal{R}_K\|_{0,K}|\boldsymbol{w}|_{1,K},\\
(\nabla\cdot\boldsymbol{u}_h,b_K r)_K &\leq \|\nabla\cdot\boldsymbol{u}_h\|_{0,K}\|r\|_{0,K},\\
(\mathcal{R}_F,b_F\mathcal{P}_F(\boldsymbol{s}))_F &\leq \|\mathcal{R}_F\|_{0,K}\|\mathcal{P}_F(\boldsymbol{s})\|_{0,F},
\end{aligned}
$$

for all $r \in \mathbb{P}_0(K)$, $\boldsymbol{w} \in \mathbb{P}_1(K)^2$ and $\boldsymbol{s} \in \mathbb{P}_l(F)^2$, with $l = 0, 1$, thus we arrive at

$$\|F(\boldsymbol{u}_h, p_h)\|_{\mathcal{B}_h'} \quad \leq \quad C \left\{ \sum_{K \in \mathcal{T}_h} \eta_K^2 \right\}^{1/2} .$$

Observing that $\|\chi_h(\boldsymbol{v} \cdot \boldsymbol{x})\|_{0,K} \leq C h_K \, \|\boldsymbol{v}\|_{0,K}$ for all $\boldsymbol{v} \in \mathbb{P}_0(K)^2$, and using (9) and method (20), we get

$$\begin{aligned}
&\langle F(\boldsymbol{u}_h, p_h) - F_h(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h) \rangle \\
\leq\;& C \Bigg[\sum_{K \in \mathcal{T}_h} \frac{\alpha_K h_K^2}{\nu} \, \|\boldsymbol{f} + \nu \, \Delta \boldsymbol{u}_h - (\nabla \boldsymbol{u}_h) \Pi_K \boldsymbol{u}_h - \nabla p_h\|_{0,K} \|\nabla q_h - (\nabla \boldsymbol{v}_h) \Pi_K \boldsymbol{u}_h\|_{0,K} \\
&\quad + \sum_{K \in \mathcal{T}_h} \frac{\gamma_K h_K^2}{\nu} \, \|\nabla \cdot \boldsymbol{u}_h\|_{0,K} \|\nabla \cdot \boldsymbol{v}_h\|_{0,K} + \sum_{F \in \mathcal{E}_h} \tau_F \|\mathcal{R}_F\|_{0,F} \|[\![\nu \, \partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!]\|_{0,F} \Bigg] \\
\leq\;& C \Bigg[\sum_{K \in \mathcal{T}_h} \frac{\alpha_K h_K^2}{\nu} \, \|\mathcal{R}_K + (\nabla \boldsymbol{u}_h) \chi_h(\boldsymbol{u}_h)\|_{0,K} \left[|q_h|_{1,K} + \|\Pi_K \boldsymbol{u}_h\|_{\infty,K} |\boldsymbol{v}_h|_{1,K} \right] \\
&\quad + \sum_{K \in \mathcal{T}_h} \frac{\gamma_K h_K^2}{\nu} \, \|\nabla \cdot \boldsymbol{u}_h\|_{0,K} |\boldsymbol{v}_h|_{1,K} + \sum_{F \in \mathcal{E}_h} \tau_F \|\mathcal{R}_F\|_{0,F} \|[\![\nu \, \partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!]\|_{0,F} \Bigg] \\
\leq\;& C \Bigg[\sum_{K \in \mathcal{T}_h} \frac{\alpha_K h_K}{\nu} \, \|\mathcal{R}_K + (\nabla \boldsymbol{u}_h) \chi_h(\boldsymbol{u}_h)\|_{0,K} \left[\|q_h\|_{0,K} + \|\boldsymbol{u}_h\|_{0,K} |\boldsymbol{v}_h|_{1,K} \right] \\
&\quad + \sum_{K \in \mathcal{T}_h} \frac{\gamma_K h_K^2}{\nu} \, \|\nabla \cdot \boldsymbol{u}_h\|_{0,K} |\boldsymbol{v}_h|_{1,K} + \sum_{F \in \mathcal{E}_h} \tau_F \|\mathcal{R}_F\|_{0,F} \|[\![\nu \, \partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!]\|_{0,F} \Bigg] \\
\leq\;& C \Lambda(\nu, \boldsymbol{u}_h) \left\{ \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{\nu} \left[\|\mathcal{R}_K\|_{0,K}^2 + \|(\nabla \boldsymbol{u}_h) \chi_h(\boldsymbol{u}_h)\|_{0,K}^2 + \frac{h_K^2}{\nu^2} \, \|\nabla \cdot \boldsymbol{u}_h\|_{0,K}^2 \right] + \sum_{F \in \mathcal{E}_h} \tau_F \|\mathcal{R}_F\|_{0,F}^2 \right\}^{1/2} \\
&\qquad \left\{ \nu \, |\boldsymbol{v}_h|_{1,\Omega}^2 + \frac{1}{\nu} \, \|q_h\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_h} \tau_F \|[\![\nu \, \partial_{\boldsymbol{n}} \boldsymbol{v}_h + q_h \boldsymbol{n}]\!]\|_{0,F}^2 \right\}^{1/2} \\
\leq\;& C \Lambda(\nu, \boldsymbol{u}_h) \left\{ \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{\nu} \left[\|\mathcal{R}_K\|_{0,K}^2 + \|(\nabla \boldsymbol{u}_h) \chi_h(\boldsymbol{u}_h)\|_{0,K}^2 + \frac{h_K^2}{\nu^2} \, \|\nabla \cdot \boldsymbol{u}_h\|_{0,K}^2 \right] + \sum_{F \in \mathcal{E}_h} \tau_F \|\mathcal{R}_F\|_{0,F}^2 \right\}^{1/2} \\
&\qquad \left\{ \nu \, |\boldsymbol{v}_h|_{1,\Omega}^2 + \frac{1}{\nu} \, \|q_h\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_h} \tau_F \nu^2 \, \|[\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h]\!]\|_{0,F}^2 + \sum_{F \in \mathcal{E}_h} \tau_F \, \|[\![q_h]\!]\|_{0,F}^2 \right\}^{1/2}, \qquad (42)
\end{aligned}$$

where $\Lambda(\nu, \boldsymbol{u}_h) := \max \left\{ 1, \frac{\|\boldsymbol{u}_h\|_{0,\Omega}}{\nu} \right\}$.

Applying mesh regularity, (10), (22), and local trace (8), we arrive at

$$
\begin{aligned}
&\sum_{F\in\mathcal{E}_h} \tau_F \nu^2 \,\|[\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h} \tau_F \,\|[\![q_h]\!]\|_{0,F}^2 \\
&\le C \left\{ \sum_{F\in\mathcal{E}_h} \nu h_F \,\|[\![\partial_{\boldsymbol{n}} \boldsymbol{v}_h]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h} \frac{h_F}{\nu} \,\|[\![q_h]\!]\|_{0,F}^2 \right\} \\
&\le C \left\{ \sum_{K\in\mathcal{T}_h} \nu h_K \,\|\partial_{\boldsymbol{n}} \boldsymbol{v}_h\|_{0,\partial K}^2 + \sum_{F\in\mathcal{E}_h} \frac{1}{\nu} \,\|q_h\|_{0,\omega_F}^2 \right\} \\
&\le C \left\{ \sum_{K\in\mathcal{T}_h} \nu\, h_K \left[h_K^{-1} \,|\boldsymbol{v}_h|_{1,K}^2 + h_K \,|\boldsymbol{v}_h|_{2,K}^2 \right] + \frac{1}{\nu} \,\|q_h\|_{0,\Omega}^2 \right\} \\
&\le C \,\|(\boldsymbol{v}_h, q_h)\|^2.
\end{aligned} \tag{43}
$$

Combining (42) with (43), it holds

$$
\|F(\boldsymbol{u}_h, p_h) - F_h(\boldsymbol{u}_h, p_h)\|_{(\mathbf{V}_h\times Q_h)'} \quad \le \quad C\,\Lambda(\nu, \boldsymbol{u}_h)\,\eta_H, \tag{44}
$$

thus from (42), (40) and (44) we obtain that

$$
\|F(\boldsymbol{u}_h, p_h)\|_{(\mathbf{V}\times Q)'} \quad \le \quad C\,\Lambda(\nu, \boldsymbol{u}_h)\,\eta_H. \tag{45}
$$

Finally, using (29) and (45) the result follows. □

5. Numerical validation

We solve RELP method (20) by a Newton–Picard scheme. The idea consists of starting with a solution $\boldsymbol{u}_h^0$, p_h^0 and perform the following:

`For` $n = 1, 2, 3, \dots$

(1) Compute $\delta\boldsymbol{u}_h^n$ and δp_h^n from the linear system

$$
\begin{aligned}
&\nu\,(\nabla\delta\boldsymbol{u}_h^n, \nabla\boldsymbol{v}_h) - (\delta p_h^n, \nabla\cdot\boldsymbol{v}_h) - (q_h, \nabla\cdot\delta\boldsymbol{u}_h^n) + ((\nabla\delta\boldsymbol{u}_h^n)\boldsymbol{u}_h^n, \boldsymbol{v}_h) + ((\nabla\boldsymbol{u}_h^n)\delta\boldsymbol{u}_h^n, \boldsymbol{v}_h) \\
&- \sum_{K\in\mathcal{T}_h} \frac{\alpha_K^n}{\nu} \Big(\chi_h(\delta p_h^n + \boldsymbol{x}\cdot(\nabla\delta\boldsymbol{u}_h^n)\Pi_K\boldsymbol{u}_h^n), \chi_h(q_h - \boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h^n) \Big)_K \\
&+ \sum_{K\in\mathcal{T}_h} \frac{\gamma_K^n}{\nu} \Big(\chi_h(\boldsymbol{x}\,\nabla\cdot\delta\boldsymbol{u}_h^n), \chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h) \Big)_K \\
&- \sum_{F\in\mathcal{E}_h} \tau_F^n \Big([\![-\nu\partial_n\delta\boldsymbol{u}_h^n + \delta p_h^n\boldsymbol{n}]\!], [\![\nu\partial_n\boldsymbol{v}_h + q_h\boldsymbol{n}]\!] \Big)_F \\
= \quad &(\boldsymbol{f}, \boldsymbol{v}_h)_\Omega - \nu\,(\nabla\boldsymbol{u}_h^n, \nabla\boldsymbol{v}_h)_\Omega + (p_h^n, \nabla\cdot\boldsymbol{v}_h)_\Omega + (q_h, \nabla\cdot\boldsymbol{u}_h^n)_\Omega - ((\nabla\boldsymbol{u}_h^n)\boldsymbol{u}_h^n, \boldsymbol{v}_h)_\Omega \\
&- \sum_{K\in\mathcal{T}_h} \frac{\alpha_K^n}{\nu} \Big(\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} - p_h^n - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h^n)\Pi_K\boldsymbol{u}_h^n), \chi_h(q_h - \boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{u}_h^n) \Big)_K \\
&- \sum_{K\in\mathcal{T}_h} \frac{\gamma_K^n}{\nu} \Big(\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{u}_h^n), \chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h) \Big)_K \\
&+ \sum_{F\in\mathcal{E}_h} \tau_F^n \Big([\![-\nu\partial_n\boldsymbol{u}_h^n + p_h^n\boldsymbol{n}]\!], [\![\nu\partial_n\boldsymbol{v}_h + q_h\boldsymbol{n}]\!] \Big)_F,
\end{aligned}
$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where $\alpha_K^n := \alpha_K(\boldsymbol{u}_h^n)$, $\gamma_K^n := \gamma_K(\boldsymbol{u}_h^n)$ and $\tau_F^n := \tau_F(\boldsymbol{u}_h^n)$.

(2) Set $\boldsymbol{u}_h^{n+1} = \boldsymbol{u}_h^n + \delta \boldsymbol{u}_h^n$.

(3) Set $p_h^{n+1} = p_h^n + \delta p_h^n$.

(4) If convergence `then exit`.

`End For`.

Next, we validate the stabilized method and the a posteriori error estimator. We first adopt a numerical test with an analytic solution, followed by some well-established benchmarks from the fluid dynamics literature. We measure the quality of the a posteriori error estimator through the so–called *effectivity index*, which is required to remain bounded as h goes to zero and is defined by

$$E := \frac{\eta_H}{\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|}.$$

5.1. **Analytic solution.** The domain is $\Omega := (0,1)\times(0,1)$ and $\nu = 1,\ 10^{-2}$, and $\boldsymbol{f}$ is chosen such that the exact solution is given by

$$u_1(x,y) := y - \frac{1-e^{y/\nu}}{1-e^{1/\nu}}, \quad u_2(x,y) := x - \frac{1-e^{x/\nu}}{1-e^{1/\nu}}, \quad p(x,y) := x-y.$$

Figures 2–5 show that method (20) remains precise when the viscosity coefficient is small. We notice that the method achieves optimal order of convergence for both pair of spaces $\mathbb{P}_1^2 \times \mathbb{P}_1$ and $\mathbb{P}_1^2 \times \mathbb{P}_0$. In Tables 1–4 we point out that the effectivity index stays bounded when h goes to zero for different values of ν.

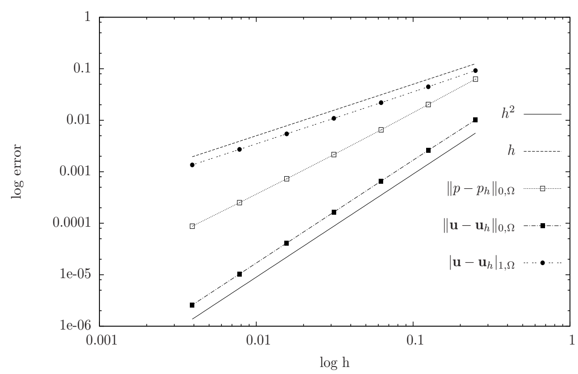

FIGURE 2. Analytic solution with $\nu = 1$. Convergence history for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element.

h	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	η_H	E
0.125	0.0490	0.2289	4.6708
0.0625	0.0229	0.1170	5.1053
0.03125	0.0111	0.0590	5.3081
0.015625	0.0055	0.0296	5.4000
0.0078125	0.0027	0.0148	5.4424
0.0039062	0.0014	0.0074	5.4625

TABLE 1. Analytic solution with $\nu = 1$. Exact error, a posteriori error estimator and effectivity index for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element.

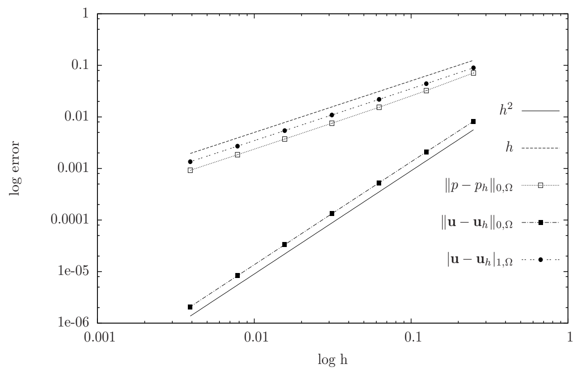

FIGURE 3. Analytic solution with $\nu = 1$. Convergence history for the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

h	$\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|$	η_H	E
0.125	0.0546	0.3077	5.6311
0.0625	0.0267	0.1553	5.8107
0.03125	0.0132	0.0780	5.8967
0.015625	0.0066	0.0391	5.9378
0.0078125	0.0033	0.0196	5.9576
0.0039062	0.0016	0.0098	5.9673

TABLE 2. Analytic solution with $\nu = 1$. Exact error, a posteriori error estimator and effectivity index for the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

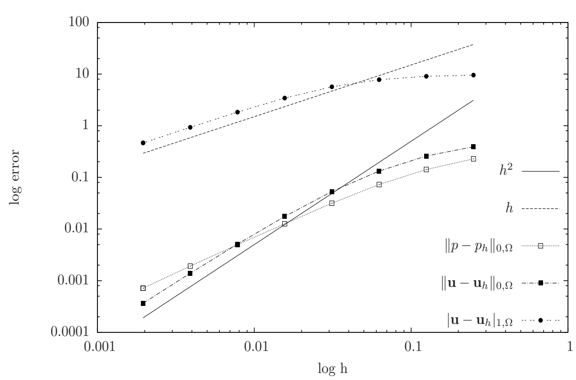

FIGURE 4. Analytic solution: $\nu = 10^{-2}$. Convergence history with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element.

h	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	η_H	E
0.125	1.6857	9.7849	5.8047
0.0625	1.0664	5.5329	5.1883
0.03125	0.6497	2.7666	4.2583
0.015625	0.3665	1.4270	3.8940
0.0078125	0.1903	0.7809	4.1032
0.0039062	0.0950	0.4171	4.3902

TABLE 3. Analytic solution with $\nu = 10^{-2}$. Exact error, a posteriori error estimator and effectivity index for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element.

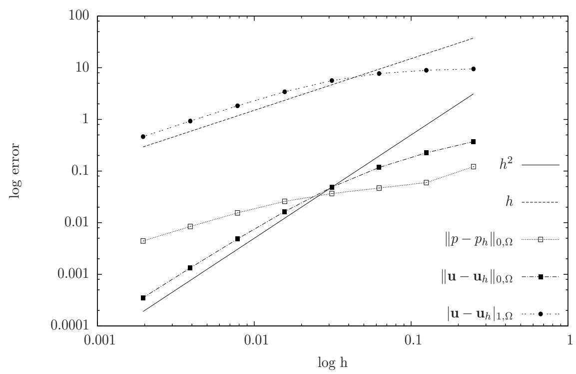

FIGURE 5. Analytic solution with $\nu = 10^{-2}$. Convergence history with the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

h	$\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|$	η_H	E
0.125	1.0715	10.9480	10.2180
0.0625	0.8689	5.9920	6.8962
0.03125	0.5991	2.8741	4.7976
0.015625	0.3538	1.4606	4.1277
0.0078125	0.1876	0.7921	4.2230
0.0039062	0.0946	0.4218	4.4570

TABLE 4. Analytic solution with $\nu = 10^{-2}$. Exact error, a posteriori error estimator and effectivity index with the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

5.2. **Lid–driven cavity problem.** The lid–driven cavity problem is a standard benchmark in computational fluid mechanics (see [19] and [30], for instance). The Reynolds number is given by $Re := \frac{1}{\nu}$, and we perform the computation assuming $Re = 5000$ and $Re = 10000$. The final adapted mesh and the streamlines of the velocity on this mesh are depicted in Figure 6, for $Re = 5000$, and in Figure 7, for $Re = 10000$. We observe that the mesh refinement concentrates inside the primary vortex which leads to a accurate approximation of the solution.

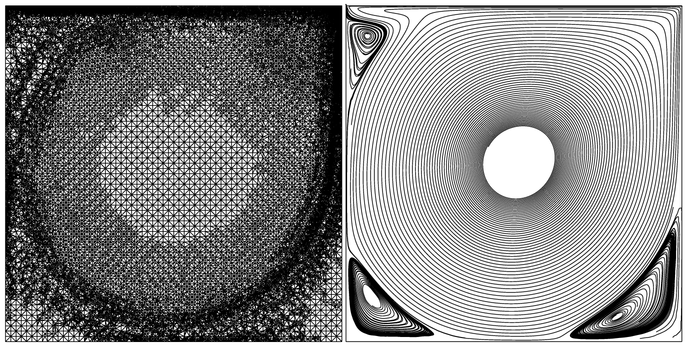

FIGURE 6. Lid–driven cavity with $Re = 5000$. Adapted mesh with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element and streamlines of the velocity (53.157 elements).

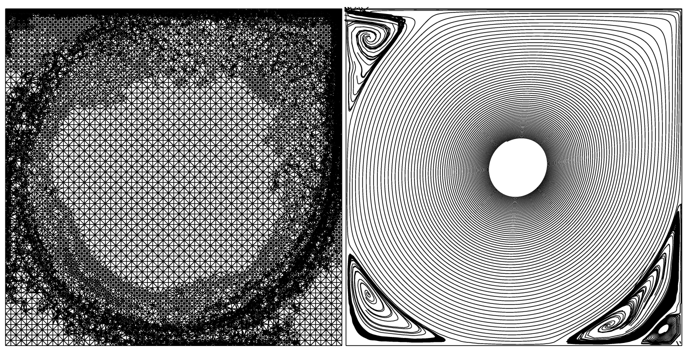

FIGURE 7. Lid–driven cavity with $Re = 10000$. Adapted mesh with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element and streamlines of the solution (51.961 elements).

Finally, Table 5 shows that the location of the center of the primary vortex using RELP method (20) is in accordance with the one obtained from Ghia and Shin in [19], and from Medic and Mohammadi in [24].

Scheme	$Re = 5.000$	$Re = 10.000$
Ghia et al. [19]	$x = 0.5117$; $y = 0.5352$	$x = 0.5117$; $y = 0.5333$
Medic et. al. [24]	$x = 0.53$; $y = 0.53$	$x = 0.525$; $y = 0.53$
RELP $\mathbb{P}_1^2 \times \mathbb{P}_1$ (adapted mesh)	$x = 0.5155$;$y = 0.5352$	$x = 0.5127$; $y = 0.5296$
RELP $\mathbb{P}_1^2 \times \mathbb{P}_0$ (adapted mesh)	$x = 0.5205$; $y = 0.5309$	$x = 0.5197$; $y = 0.5238$

TABLE 5. Lid–driven cavity. Position of the center of the primary vortex.

5.3. **Backward facing step problem.** This test case is posed on a backward facing step configuration (see [23]). The step starts at $(x, y) = (0, 0)$, the entry of the channel is at $x = -5$ and the exit at $x = 20$. The channel width is $h_1 = 1$ at the entry and $H = 2$ at the exit. We prescribe at the inflow a parabolic profile

$\boldsymbol{u}_p = (4y(1-y),0)^T$ and a free flow condition at outflow. Also, we assume $\boldsymbol{f} = \boldsymbol{0}$. The Reynolds number here is defined by $Re := \dfrac{\bar{u}_1 H}{\nu} = 800$, where the mean velocity $\bar{u}_1$ is equal to $2/3$. We recall that a singularity on the solution arises induced by the re–entrant corner.

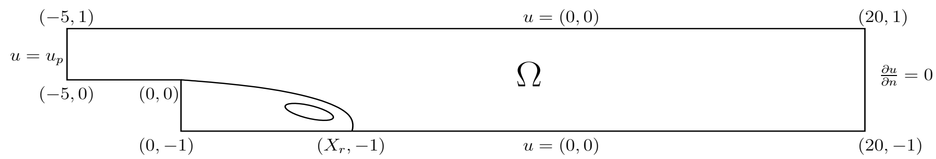

FIGURE 8. Boundary conditions for the problem.

A zoom of the final adapted mesh is presented for Figure 9 in the $\mathbb{P}_1^2 \times \mathbb{P}_0$ case. In Figure 10 we plot the isovalues of $|\boldsymbol{u}_h|$ and the streamlines of the velocity for the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

Now, being X_r the distance from the step to the lower attachment point (see Figure 8) we define the reattachment length by X_r/H. In Table 6, we compare such a quantity to the experimental result given in [6] and also to the numerical solution from [23]. We observe that all numerical solutions underestimate the reattachment length with respect to the experimental result which is probably due to three dimensional effects (see [23], [24]).

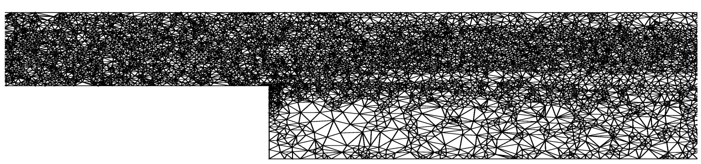

FIGURE 9. The backward facing step problem $Re = 800$. Zoom of the adapted mesh (63.506 elements) with the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

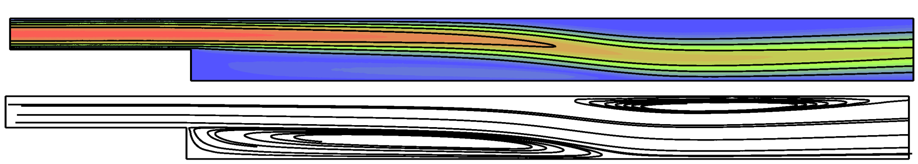

FIGURE 10. The backward facing step problem $Re = 800$. Isovalues $|\boldsymbol{u}_h|$ (top) and the streamlines of the velocity (bottom) using the final adapted mesh (63.506 elements) with the $\mathbb{P}_1^2 \times \mathbb{P}_0$ element.

Scheme	X_r/H
Armaly et. al. [6]	7.1
Lê et. al. [23]	5.95
RELP $\mathbb{P}_1^2/\mathbb{P}_1$ (adapted mesh)	5.95
RELP $\mathbb{P}_1^2/\mathbb{P}_0$ (adapted mesh)	5.86

TABLE 6. The backward facing step problem. Reattachment length for $Re = 800$.

5.4. **Circular cylinder problem.** The domain and the boundary conditions are shown in Figure 11. The inflow velocity field is $\boldsymbol{u}_p = (1.2\,y\,(0.41 - y)/0.41^2, 0)^T = (U, 0)$ and the viscosity is set to $\nu = 10^{-3}$ (for further details, see [30]).

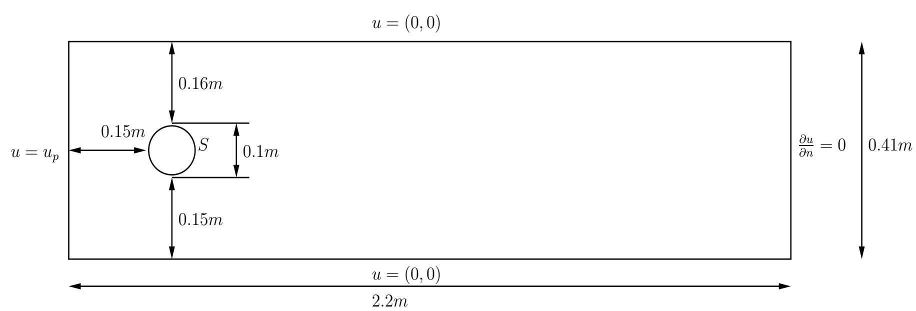

FIGURE 11. Boundary conditions for the problem.

The *drag* and *lift* coefficients are useful to validate numerical schemes, and are defined by

$$C_D := \frac{2}{\bar{u}^2 D} \int_S (\nu \, \frac{\partial v_t}{\partial \boldsymbol{n}} \, n_y - P n_x) \, dS, \qquad C_L := -\, \frac{2}{\bar{u}^2 D} \int_S (\nu \, \frac{\partial v_t}{\partial \boldsymbol{n}} \, n_x + P n_y) \, dS,$$

where we used the following notations: S corresponds to the boundary of the cylinder, $\boldsymbol{n} := (n_x, n_y)$ and $\boldsymbol{t} = (n_y, -n_x)$ are, respectively, the outward normal vector and the tangent vector on S and v_t is the tangential velocity on S. The diameter of the cylinder D is set to 0.1 and the mean velocity $\bar{u}$ is $\frac{2}{3}U(0, 0.205)$.

The length of the recirculation and the difference of the pressure at points $(x_a, y_a) = (0.15, 0.2)$ and $(x_e, y_e) = (0.25, 0.2)$ are denoted by

$$L_r := x_r - x_e, \qquad \Delta p := P(x_a, y_a) - P(x_e, y_e),$$

where x_r is the x-coordinate of the end of the recirculation area. In Table 7, we compare these quantities using RELP method (20) to the ones obtained from [24] and [26]. Figure 12 depicts a zoom of the final adapted mesh with the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element. A zoom of the streamlines of the velocity and the isovalues of $|\boldsymbol{u}_h|$ are presented in Figure 13 for the adapted mesh.

Scheme	C_D	C_L	Δp	L_r
Schäfer et. al. [26]	5.58	0.011	0.1175	0.085
Medic et. al. [24]	5.65	0.012	0.121	0.082
RELP $\mathbb{P}_1^2 \times \mathbb{P}_1$ (adapted mesh)	5.56	0.010	0.1170	0.084
RELP $\mathbb{P}_1^2 \times \mathbb{P}_0$ (adapted mesh)	5.54	0.012	0.1171	0.084

TABLE 7. The circular cylinder problem with $\nu = 10^{-3}$.

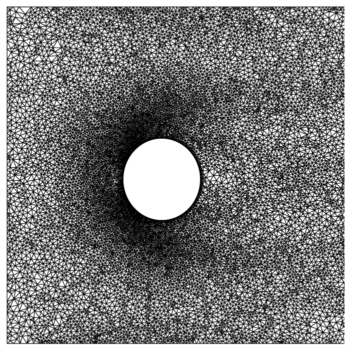

FIGURE 12. The circular cylinder problem with $\nu = 10^{-3}$. Zoom adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (60.593 elements).

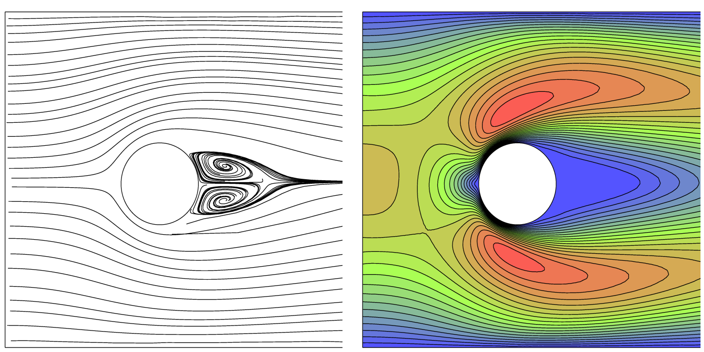

FIGURE 13. The circular cylinder problem with $\nu = 10^{-3}$. Zoom of the streamlines and the isolines of $|\boldsymbol{u}_h|$ in the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (60.593 elements).

5.5. **The flat plate problem.** Concerning a laminar flow over a flat plate, closed formulas for the friction coefficient and for the velocity profile are available to comparisons (see Blasius [27]). The statement of this problem follows [24] and consists of a rectangular domain $\Omega := (-0.2, 1) \times (0, 0.1)$ with prescribed velocity $\boldsymbol{u}_p = (1, 0)^T$ at inflow boundary and viscosity $\nu = \frac{1}{33000}$ (i.e. $Re = 33000$), and $\boldsymbol{f} = \mathbf{0}$. Since non-slip condition is imposed on the flat plate, a boundary layer starts at the "border of attack" and may be considered fully developed after a short distance.

Figure 14 depicts a zoom of the final adapted mesh using both pairs of interpolation spaces. As a result, we observe a dense concentration of elements inside the boundary layer region.

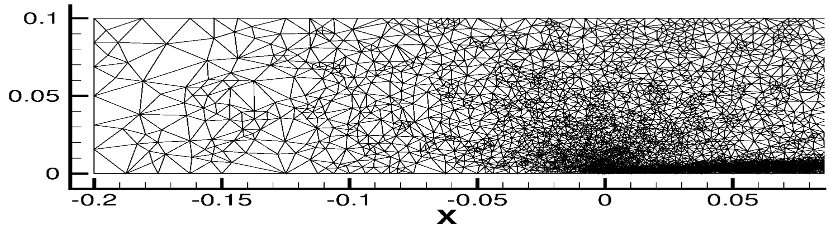

Figure 14. The flat plate problem with $Re = 33000$. A zoom of the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ case (95.099 elements).

We compare the friction coefficient $c_f := \nu \dfrac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}} \cdot \boldsymbol{t}$ in Figure 15, as well as the profile of the horizontal velocity at $x = 0.2$ with Blasius' solution. Here $\boldsymbol{t}$ is the unit tangent vector on the plate. Figure 16 shows the isovalues of $|\boldsymbol{u}_h|$ and the isolines of the pressure for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element. We notice the absence of numerical spurious oscillations at the vicinity of the boundary layer which highlights the robustness of the approach.

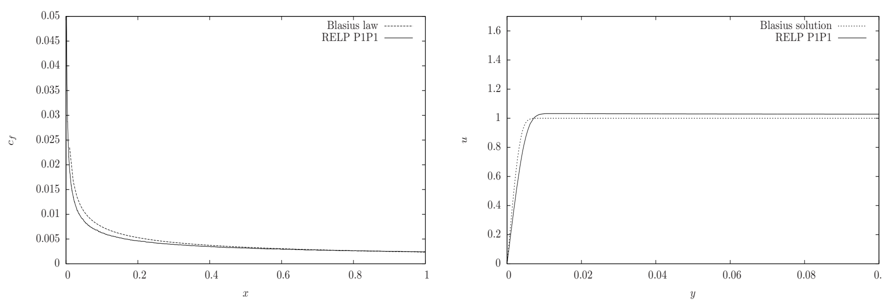

Figure 15. Comparison of friction coefficient c_f on the plate (left) and a profile of the horizontal velocity at $x = 0.2$ (right) to Blasius solution.

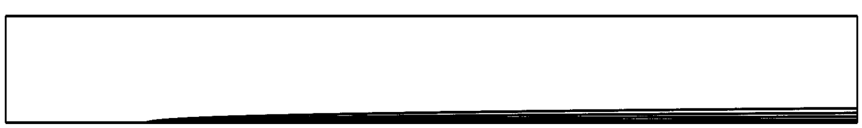

Figure 16. The flat plate problem with $Re = 33000$. Isovalues of $|\boldsymbol{u}_h|$ computed on the final adapted mesh for the $\mathbb{P}_1^2 \times \mathbb{P}_1$ element (95.099 elements).

6. Conclusions

We have presented a new a posteriori error estimator for the fully non-linear Navier-Stokes equations which efficiently drives mesh adaptation. We proved the estimator is equivalent to the true errors in natural norms. Also, the stabilized method used to construct the estimator is proved to be well-posed using a fixed point theory. Extensive numerical validations attested the accuracy of the methodology when approximating high Reynolds number flows on a large variety of geometries.

Appendix A. Uniqueness of the discrete solution

We prove a uniqueness result for method (20) under the diffusion dominated assumption. As such, we set $\alpha_K = \gamma_K = 1$ for all $K \in \mathcal{T}_h$, and assume that $\tau_F = \dfrac{h_F}{12\nu}$ for all $F \in \mathcal{E}_h$ since both expressions in (22) are equivalent in this regime (for details see Lemma 2 in [11]). Thereby, under such simplifications, method (20) reads: *Find* $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *such that*

$$
\begin{aligned}
&\nu\,(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) - (p_h, \nabla\cdot \boldsymbol{v}_h) - (q_h, \nabla\cdot \boldsymbol{u}_h)\\
&- \frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\cdot(\nabla \boldsymbol{u}_h)\,\Pi_K \boldsymbol{u}_h + p_h), \chi_h(-\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\,\Pi_K \boldsymbol{u}_h + q_h))_K\\
&+ \frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h), \chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{v}_h))_K - \sum_{F\in\mathcal{E}_h} \frac{h_F}{12\,\nu}\,([\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + p_h\boldsymbol{n}]\!], [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\boldsymbol{n}]\!])_F\\
&= (\boldsymbol{f}, \boldsymbol{v}_h) - \frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\cdot\Pi_K \boldsymbol{f}), \chi_h(-\boldsymbol{x}\cdot(\nabla \boldsymbol{v}_h)\,\Pi_K \boldsymbol{u}_h + q_h)))_K \quad \forall(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \qquad (46)
\end{aligned}
$$

We first rewrite (46) as a fixed point problem. To this end, we define $T_h : \mathbf{V}' \times Q \longrightarrow \mathbf{V}_h \times Q_h$ the discrete Stokes operator, which for each $(\boldsymbol{w}, r) \in \mathbf{V}' \times Q$, it associates the unique solution $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of

$$
\mathbf{A}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \langle \boldsymbol{w}, \boldsymbol{v}_h\rangle + (r, q_h), \qquad (47)
$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where $\boldsymbol{A}(\cdot,\cdot)$ reads

$$
\begin{aligned}
&\boldsymbol{A}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) :=\\
&\nu\,(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla\cdot \boldsymbol{v}_h) - (q_h, \nabla\cdot \boldsymbol{u}_h) - \frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(p_h), \chi_h(q_h))_K\\
&+\frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{u}_h), \chi_h(\boldsymbol{x}\,\nabla\cdot \boldsymbol{v}_h))_K - \sum_{F\in\mathcal{E}_h} \frac{h_F}{12\,\nu}\,([\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + p_h\boldsymbol{n}]\!], [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\boldsymbol{n}]\!])_F\,. \qquad (48)
\end{aligned}
$$

Also, we introduce the mapping $G_h : H^2(\mathcal{T}_h)^2 \times H^1(\mathcal{T}_h) \longrightarrow \mathbf{V}_h \times Q_h$ where $(\boldsymbol{w}_h, r_h) := G_h(\boldsymbol{z}, t)$ solves

$$
\begin{aligned}
(\boldsymbol{w}_h, \boldsymbol{v}_h) + (r_h, q_h) &= -(\boldsymbol{f} - (\nabla \boldsymbol{z})\boldsymbol{z}, \boldsymbol{v}_h) + \frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} - \boldsymbol{x}\cdot(\nabla\boldsymbol{z})\Pi_K\boldsymbol{z}), \chi_h(q_h))_K\\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h} (\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} + \boldsymbol{x}\cdot\nu\Delta\boldsymbol{z} - \boldsymbol{x}\cdot(\nabla\boldsymbol{z})\Pi_K\boldsymbol{z} - t), \chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{z}))_K\,,
\end{aligned}
$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. Combining these operators, problem (46) is written as the following fixed point problem

$$
-T_h G_h(\boldsymbol{u}_h, p_h) = (\boldsymbol{u}_h, p_h). \qquad (49)
$$

Before proving the uniqueness result for (20), we need to establish the well-posedness of (47). This result is presented in the next lemma.

Lemma 9. *The mapping* T_h *is well–defined.*

Proof. Defining the mesh-dependent norm

$$
\|(\boldsymbol{v}_h, q_h)\|_h^2 := \nu|\boldsymbol{v}_h|_{1,\Omega}^2 + \frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\|\chi_h(q_h)\|_{0,K}^2 + \frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\|\chi_h(\boldsymbol{x}\,\nabla\cdot\boldsymbol{v}_h)\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h}\frac{h_F}{12\nu}\,\|[\![-\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\boldsymbol{n}]\!]\|_{0,F}^2,
$$

we realize that, for all $(\boldsymbol{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, it holds

$$\boldsymbol{A}((\boldsymbol{v}_h, q_h), (\boldsymbol{v}_h, -q_h)) = \|(\boldsymbol{v}_h, q_h)\|_h^2 \,, \tag{50}$$

and, thus, the problem (47) is well–posed and the operator T_h is well–defined. □

Lemma 10. *The operator T_h is continuous. More precisely, there exists $C > 0$, independent of h and ν, such that*

$$\|T_h(\boldsymbol{w}, r)\| \leq C \sqrt{\nu}\, (1+h)^2 \, \|(\boldsymbol{w}, r)\|_{(\mathbf{V}_h \times Q_h)'} \,,$$

for all $(\boldsymbol{w}, r) \in (\mathbf{V} \times Q)'$.

Proof. The proof follows standard arguments, but we present them here for sake of completeness. Let $(\boldsymbol{u}_h, p_h) := T_h(\boldsymbol{w}, r)$. From (50) we see that

$$\|(\boldsymbol{u}_h, p_h)\|_h^2 = \boldsymbol{A}((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, -p_h)) = \langle \boldsymbol{w}, \boldsymbol{u}_h \rangle - (r, p_h) \leq \|\boldsymbol{w}\|_{\mathbf{V}_h'} |\boldsymbol{u}_h|_{1,\Omega} + \|r\|_{Q_h'} \|p_h\|_{0,\Omega} \,. \tag{51}$$

To bound the $L^2(\Omega)$-norm of p_h, let $\boldsymbol{z} \in H_0^1(\Omega)^2$ be such that

$$\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} \leq (p_h, \nabla \cdot \boldsymbol{z}) \,, \tag{52}$$

and let $\boldsymbol{z}_h$ be the Clément interpolate of $\boldsymbol{z}$. Then, integrating by parts, using that $(\boldsymbol{u}_h, p_h)$ is the solution of (47), (6) and (7), we arrive at

$$\begin{aligned}
&\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} \leq (p_h, \nabla \cdot (\boldsymbol{z} - \boldsymbol{z}_h)) + (p_h, \nabla \cdot \boldsymbol{z}_h) \\
&= - \sum_{K \in \mathcal{T}_h} (\nabla p_h, \boldsymbol{z} - \boldsymbol{z}_h)_K + \sum_{F \in \mathcal{E}_h} ([\![p_h \boldsymbol{n}]\!], \boldsymbol{z} - \boldsymbol{z}_h)_F + \nu (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{z}_h) \\
&\quad + \frac{1}{\nu} \sum_{K \in \mathcal{T}_h} (\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{u}_h), \chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h))_K - \sum_{F \in \mathcal{E}_h} \frac{h_F}{12\nu} ([\![-\nu \partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \boldsymbol{n}]\!], [\![\nu \partial_{\boldsymbol{n}} \boldsymbol{z}_h]\!])_F - \langle \boldsymbol{w}, \boldsymbol{z}_h \rangle \\
&\leq C \sum_{K \in \mathcal{T}_h} h_K \|\nabla p_h\|_{0,K} |\boldsymbol{z}|_{1,\omega_K} + C \sum_{F \in \mathcal{E}_h} h_F^{1/2} \|[\![p_h \boldsymbol{n}]\!]\|_{0,F} |\boldsymbol{z}|_{1,\omega_F} + \nu |\boldsymbol{u}_h|_{1,\Omega} |\boldsymbol{z}_h|_{1,\Omega} + \|\boldsymbol{w}\|_{\mathbf{V}_h'} |\boldsymbol{z}_h|_{1,\Omega} \\
&\quad + \frac{C}{\nu} \sum_{K \in \mathcal{T}_h} \|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{u}_h)\|_{0,K} \|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h)\|_{0,K} + \sum_{F \in \mathcal{E}_h} \frac{h_F}{12\nu} \|[\![-\nu \partial_{\boldsymbol{n}} \boldsymbol{u}_h + p_h \boldsymbol{n}]\!]\|_{0,F} \|[\![\nu \partial_{\boldsymbol{n}} \boldsymbol{z}_h]\!]\|_{0,F} \,.
\end{aligned} \tag{53}$$

Next, using the generalized Poincaré's inequality and the fact that $|\boldsymbol{x}|_{1,K} \leq C h_K$ and (6) we obtain

$$\|\chi_h(\boldsymbol{x} \nabla \cdot \boldsymbol{z}_h)\|_{0,K} = \frac{\|\chi_h(\boldsymbol{x})\|_{0,K}}{|K|^{1/2}} \|\nabla \cdot \boldsymbol{z}_h\|_{0,K} \leq C h_K \, \|\boldsymbol{z}\|_{1,\omega_K} \,, \tag{54}$$

and then from (10), (11), (53) and the mesh regularity, we get

$$\begin{aligned}
&\beta \, \|p_h\|_{0,\Omega} |\boldsymbol{z}|_{1,\Omega} \\
&\leq C \left\{ \frac{1}{\nu} \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{h_F}{\nu} \|[\![p_h]\!]\|_{0,F}^2 + \|(\boldsymbol{u}_h, p_h)\|_h^2 + \frac{1}{\nu} \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 \right\}^{1/2} \\
&\quad \left\{ \nu |\boldsymbol{z}|_{1,\Omega}^2 + \frac{1}{\nu} \sum_{K \in \mathcal{T}_h} h_K^2 \|\boldsymbol{z}\|_{1,\omega_K}^2 \right\}^{1/2} \\
&\leq \, C \sqrt{\nu} \left\{ \|(\boldsymbol{u}_h, p_h)\|_h^2 + \frac{1}{\nu} \|\boldsymbol{w}\|_{\mathbf{V}_h'}^2 \right\}^{1/2} (1+h) \, |\boldsymbol{z}|_{1,\Omega} \,.
\end{aligned} \tag{55}$$

Now, dividing the above inequality by $\sqrt{\nu}|\boldsymbol{z}|_{1,\Omega}$, it holds

$$\frac{1}{\sqrt{\nu}}\|p_h\|_{0,\Omega} \leq C\,(1+h)\,\left\{\|(\boldsymbol{u}_h,p_h)\|_h^2 + \frac{1}{\nu}\|\boldsymbol{w}\|^2_{\mathbf{V}_h'}\right\}^{1/2}. \tag{56}$$

Next, using (56) in (51), and $a\,b \leq a^2 + \frac{1}{4}\,b^2$ with $a,\, b \in \mathbb{R}^+$, we arrive at

$$\|(\boldsymbol{u}_h,p_h)\|_h^2 \leq C\left[\|\boldsymbol{w}\|^2_{\mathbf{V}_h'} + \nu(1+h)^2\|r\|^2_{Q_h'}\right] \leq C\,\nu(1+h)^2\left[\|\boldsymbol{w}\|^2_{\mathbf{V}_h'} + \|r\|^2_{Q_h'}\right], \tag{57}$$

and then replacing this in (56) we get

$$\frac{1}{\sqrt{\nu}}\|p_h\|_{0,\Omega} \leq C\,\sqrt{\nu}(1+h)^2\left[\|\boldsymbol{w}\|_{\mathbf{V}_h'} + \|r\|_{Q_h'}\right.$$

Finally, the proof ends observing that $\|(\boldsymbol{u}_h,p_h)\| \leq \left\{\|(\boldsymbol{u}_h,p_h)\|_h^2 + \frac{1}{\nu}\|p_h\|^2_{0,\Omega}\right\}^{1/2}$ and using $\|\boldsymbol{w}\|_{\mathbf{V}_h'} + \|r\|_{Q_h'} \leq C\,\|(\boldsymbol{w},r)\|_{(\mathbf{V}_h\times Q_h)'}$. □

We are ready to prove the uniqueness result. We recall that that ν is assumed to be large enough so that (20) reduces to (46). Let $(\boldsymbol{u}_h,p_h) \in \mathbf{V}_h \times Q_h$ be a solution of (46), and observe that from (49), $(\boldsymbol{u}_h,p_h)$ corresponds to a fixed point of the operator $-T_hG_h$. The proof then reduces to prove that the operator $-T_hG_h$ is a strict contraction in $B := \{(\boldsymbol{v}_h,q_h) \in \mathbf{V}_h \times Q_h \;:\; \|(\boldsymbol{v}_h,q_h)\| \leq 1\}$ and then the result follows from Banach's fixed point Theorem.

Let $(\boldsymbol{u}_h,p_h)$, $(\boldsymbol{v}_h,q_h) \in B$. Using Lemma 10 and the definition of operators T_h and G_h, it holds

$$\begin{aligned}
&\|T_hG_h(\boldsymbol{u}_h,p_h) - T_hG_h(\boldsymbol{v}_h,q_h)\| = \|T_h(G_h(\boldsymbol{u}_h,p_h) - G_h(\boldsymbol{v}_h,q_h))\| \\
&\leq C\sqrt{\nu}(1+h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\leq 1} (G_h(\boldsymbol{u}_h,p_h) - G_h(\boldsymbol{v}_h,q_h),(\boldsymbol{w}_h,t_h)) \\
&\leq C\sqrt{\nu}(1+h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\leq 1} \Big\{((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla\boldsymbol{v}_h)\boldsymbol{v}_h,\boldsymbol{w}_h) \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - p_h),\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K\boldsymbol{u}_h))_K \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(-\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} + \boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K.\boldsymbol{v}_h + q_h),\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K\boldsymbol{v}_h))_K \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h),\chi_h(t_h))_K\Big\} \\
&\leq C\sqrt{\nu}(1+h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\leq 1} \Big\{((\nabla\boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla\boldsymbol{v}_h)\boldsymbol{v}_h,\boldsymbol{w}_h) \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - p_h),\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K(\boldsymbol{u}_h - \boldsymbol{v}_h)))_K \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(-\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - (\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h) - (p_h - q_h),\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K\boldsymbol{v}_h))_K \\
&-\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}(\chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h),\chi_h(t_h))_K\Big\} \\
&= C\sqrt{\nu}(1+h)^2 \sup_{\|(\boldsymbol{w}_h,t_h)\|\leq 1} \Big\{\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\Big\}.
\end{aligned} \tag{58}$$

Regarding item I above, we use that $((\nabla\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}) \leq C\,|\boldsymbol{u}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega}|\boldsymbol{w}|_{1,\Omega}$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{V}$, and the definition of the norm $\|\cdot\|$ to get

$$
\begin{aligned}
\mathrm{I} &= ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)\boldsymbol{v}_h, \boldsymbol{w}_h) \\
&= ((\nabla(\boldsymbol{u}_h - \boldsymbol{v}_h))\boldsymbol{u}_h - (\nabla \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{u}_h), \boldsymbol{w}_h) \\
&\leq C\left\{|\boldsymbol{u}_h - \boldsymbol{v}_h|_{1,\Omega}|\boldsymbol{u}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega} + |\boldsymbol{v}_h|_{1,\Omega}|\boldsymbol{u}_h - \boldsymbol{v}_h|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega}\right\} \\
&\leq \frac{C}{\nu\sqrt{\nu}}\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\|. \qquad (59)
\end{aligned}
$$

To bound item II, we employ (9), (11), Hölder and Cauchy–Schwarz inequalities, and the definition of the norm $\|\cdot\|$ as follows

$$
\begin{aligned}
\mathrm{II} &= -\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\left(\chi_h(\boldsymbol{x}\cdot\Pi_K\boldsymbol{f} - \boldsymbol{x}\cdot(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - p_h), \chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K(\boldsymbol{u}_h - \boldsymbol{v}_h))\right)_K \\
&\leq \frac{C}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\|\boldsymbol{f} - (\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - \nabla p_h\|_{0,K}\|(\nabla\boldsymbol{w}_h)\Pi_K(\boldsymbol{u}_h - \boldsymbol{v}_h)\|_{0,K} \\
&\leq \frac{C}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\left\{\|\boldsymbol{f}\|_{0,K} + \|\nabla\boldsymbol{u}_h\|_{0,K}\|\Pi_K\boldsymbol{u}_h\|_{\infty,K} + \|\nabla p_h\|_{0,K}\right\}\|\nabla\boldsymbol{w}_h\|_{\infty,K}\|\Pi_K(\boldsymbol{u}_h - \boldsymbol{v}_h)\|_{0,K} \\
&\leq \frac{C}{\nu}\sum_{K\in\mathcal{T}_h}\left\{h_K\|\boldsymbol{f}\|_{0,K} + \|\nabla\boldsymbol{u}_h\|_{0,K}\|\boldsymbol{u}_h\|_{0,K} + \|p_h\|_{0,K}\right\}|\boldsymbol{w}_h|_{1,K}\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{0,K} \\
&\leq \frac{C}{\nu^2}\left\{h\|\boldsymbol{f}\|_{0,\Omega} + \frac{1}{\nu}\|(\boldsymbol{u}_h, p_h)\|^2 + \sqrt{\nu}\|(\boldsymbol{u}_h, p_h)\|\right\}\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\| \\
&\leq \frac{C}{\nu^2}\left\{h\|\boldsymbol{f}\|_{0,\Omega} + \frac{1}{\nu} + \sqrt{\nu}\right\}\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\|. \qquad (60)
\end{aligned}
$$

The terms III and IV are bounded using similar arguments as the ones used for II, and then we get

$$
\begin{aligned}
\mathrm{III} &= -\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\left(\chi_h(-\boldsymbol{x}\cdot((\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - (\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h) - (p_h - q_h)), \chi_h(\boldsymbol{x}\cdot(\nabla\boldsymbol{w}_h)\Pi_K\boldsymbol{v}_h)\right)_K \\
&\leq \frac{C}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\|(\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h - (\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h + \nabla(p_h - q_h)\|_{0,K}\|\Pi_K\boldsymbol{v}_h\|_{\infty,K}|\boldsymbol{w}_h|_{1,K} \\
&\leq \frac{C}{\nu^2\sqrt{\nu}}\left\{\frac{1}{\nu}(\|(\boldsymbol{u}_h, p_h)\| + \|(\boldsymbol{v}_h, q_h)\|) + \sqrt{\nu}\right\}\|(\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\| \\
&\leq \frac{C}{\nu^2\sqrt{\nu}}\left\{\frac{2}{\nu} + 1\right\}\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\|, \qquad (61)
\end{aligned}
$$

and

$$
\mathrm{IV} = -\frac{1}{\nu}\sum_{K\in\mathcal{T}_h}\left(\chi_h((\nabla\boldsymbol{v}_h)\Pi_K\boldsymbol{v}_h - (\nabla\boldsymbol{u}_h)\Pi_K\boldsymbol{u}_h), \chi_h(\nabla t_h)\right)_K \leq \frac{C}{\nu^2}\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|\|(\boldsymbol{w}_h, t_h)\|. \qquad (62)
$$

Collecting the bounds (59), (60), (61) and (62), inequality (58) becomes

$$
\|T_hG_h(\boldsymbol{u}_h, p_h) - T_hG_h(\boldsymbol{v}_h, q_h)\| \leq \frac{C}{\nu}\left\{2 + \frac{5}{\sqrt{\nu}} + \frac{h}{\sqrt{\nu}}\|\boldsymbol{f}\|_{0,\Omega}\right\}(1+h)^2\,\|(\boldsymbol{u}_h, p_h) - (\boldsymbol{v}_h, q_h)\|,
$$

and, thus, the result follows under the assumption that ν is such that $\frac{C}{\nu}\left\{2 + \frac{5}{\sqrt{\nu}} + \frac{h}{\sqrt{\nu}}\right\}(1+h)^2 < 1$.

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CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

E-mail address: raraya@ing-mat.udec.cl

Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile

E-mail address: apoza@ucsc.cl

Departamento de Matemática Aplicada e Computacional, Laboratório Nacional de Computação Científica, Av. Getúlio Vargas, 333, 25651-070 Petrópolis - RJ, Brazil

E-mail address: valentin@lncc.br