New fully-mixed finite element methods for the Stokes-Darcy coupling*

Jessika Camaño[†] Gabriel N. Gatica[‡] Ricardo Oyarzúa[§]
Ricardo Ruiz-Baier[¶] Pablo Venegas–Tapia[‖]

Abstract

In this paper we introduce and analyze two new fully-mixed variational formulations for the coupling of fluid flow with porous media flow. Flows are governed by the Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We first extend recent related results involving a pseudostress/velocity-based formulation in the fluid, and consider a fully-mixed formulation in which the main unknowns are given now by the stress, the vorticity, and the velocity, all them in the fluid, together with the velocity and the pressure in the porous medium. The aforementioned formulation is then partially augmented by introducing the Galerkin least-squares type terms arising from the constitutive and equilibrium equations of the Stokes equation, and from the relation defining the vorticity in terms of the free fluid velocity. These three terms are multiplied by stabilization parameters that are chosen in such a way that the resulting continuous formulation becomes well-posed. The classical Babuška-Brezzi theory is applied to provide sufficient conditions for the well-posedness of the continuous and discrete formulations of both approaches. Next, we derive a reliable and efficient residual-based a posteriori error estimator for the augmented mixed finite element scheme. The proof of reliability makes use of the global inf-sup condition, Helmholtz decomposition, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. In turn, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools to prove the efficiency of the estimator. Finally, several numerical results illustrating the good performance of both methods, confirming the aforementioned properties of the estimator, and showing the behaviour of the associated adaptive algorithm, are provided.

Key words: mixed finite element, Stokes equation, Darcy equation, augmented formulation, a posteriori error analysis

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[†]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `jecamano@ucsc.cl`

[‡]CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `ggatica@ci2ma.udec.cl`

[§]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `royarzua@ubiobio.cl`

[¶]Institute of Earth Sciences, Quartier UNIL-Mouline, Bâtiment Géopolis, University of Lausanne, CH-1015 Lausanne, Switzerland, e-mail: `ricardo.ruizbaier@unil.ch`

[‖]Department of Mathematics, University of Maryland, College Park, MD20742, USA, email: `pvenegas@umd.edu`

1 Introduction

The derivation of suitable numerical methods for the coupling of fluid flow (modelled by the Stokes equation) with porous media flow (modelled by the Darcy equation), has been increasing lately (see e.g., [9, 16, 18, 19, 20, 21, 24, 34, 37, 41, 42, 45, 46, 48, 50], and the references therein). The above list includes porous media with cracks, and the incorporation of other linear and nonlinear equations in the coupled problem, such as Brinkman and Forchheimer. The relevance that this model has gained through the last years, and the reason why the numerical analysis community has been putting so much effort in developing more accurate and efficient methods for solving this problem, is due to its applicability in different areas of interest, such as chemical and petroleum engineering, hydrology, and environmental sciences, to name a few.

The first fully-mixed finite element method for the 2D Stokes-Darcy coupled problem has been introduced and analyzed recently in [35]. This approach allows the introduction of further unknowns of physical interest as well as the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. Moreover, it considers dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, as the main unknowns. The pressure and the gradient of the velocity in the fluid can then be computed through a very simple post-process of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, due to the mixed structure utilized, the transmission conditions become essential, and hence they have to be imposed weakly, which leads to the incorporation of two additional unknowns to the system, namely the traces of the Darcy pressure and the Stokes velocity on the coupling interface Σ. These new unknowns are also variables of importance from a physical point of view. Then, the well known Fredholm and Babuška-Brezzi theories are applied to prove the unique solvability of the resulting continuous formulation and to derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. Among the several different ways in which the equations and unknowns can be ordered, it is chosen the one yielding a doubly mixed structure for which the inf-sup conditions of the off-diagonal bilinear forms follow straightforwardly. Moreover, the arguments of the continuous analysis can be easily adapted to the discrete case. In particular, a feasible choice of subspaces is given by Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the additional unknowns on the interface.

Furthermore, complementing the approach provided in [35], a reliable and efficient residual-based a posteriori error estimator for the fully-mixed finite element method proposed in [35] has been introduced and analyzed in [36]. The proof of reliability makes use of the global inf-sup condition, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator.

On the other hand, it is well known that when Neumann-type boundary conditions are imposed for the Stokes problem, like slip boundary conditions, the non-standard pseudostress-velocity formulation has no longer a physical meaning, and therefore a stress-velocity formulation has to be utilized instead. The latter yields a symmetry requirement for the stress tensor, which constitutes the main drawback of this kind of formulations. In fact, the difficulty in deriving and using finite element subspaces of symmetric tensors in the Stokes and Lamé systems is already well known (see, e.g. [3] and [12]). In order to circumvent these disadvantages, one can proceed as in [1], and impose the symmetry of the stress in a weak sense through the introduction of a suitable Lagrange multiplier (rotation in elasticity and vorticity in fluid mechanics), which, in the case of the Stokes system, leads to a stress-vorticity-velocity formulation. Among the different approaches for approximating the unknowns of the corresponding formulation for the Lamé system, we mention in particular the family of finite elements subspaces presented in [47], which includes the classical PEERS element from [1], and a modification of the BDM_k spaces (see [10, 11, 12]). In turn, the hypotheses on the discrete subspaces are relaxed in [26] through the introduction of a new augmented mixed formulation for linear elasticity, which allows the utilization of a RT_0–P_1–P_0 approximation for the respective three unknowns. The approach in [26], which can be easily adapted to the Stokes system (see [22], [23]), is based on the introduction of the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and from the relation connecting the rotation with the displacement.

Now, going back to our coupled problem, we recall that, because of the transmission conditions imposed on the coupling boundary, the constitutive equation of the Stokes law, defining the Stokes-Darcy coupled system, is originally written in terms of the stress tensor (see e.g. [7, 19, 42, 40]), which is certainly more realistic from a physical point of view. Motivated by this fact, in the present work we first generalize the results developed in [35] and [36] and analyze a fully-mixed variational formulation for the original coupled problem, where the main unknowns are given by the stress, the vorticity and the velocity in the fluid, together with the velocity and the pressure in the porous medium. As in [35], we apply the Babuška-Brezzi theory to prove the unique solvability of the resulting continuous formulation and to derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. Next, in order to have more flexibility in the choice of the discrete subspaces, we enrich the equations in the fluid with redundant Galerkin-type terms arising from the constitutive and equilibrium equations of the Stokes system, and from the relation connecting the vorticity with the velocity, all them multiplied by suitable stabilization parameters, so that an augmented mixed-FEM for the coupled problem is obtained. We then combine the results in [35] and [26] to prove existence and uniqueness of solution of the resulting augmented scheme, and to derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. In addition, following the approaches in [6] and [36], we develop a reliable and efficient residual-based a posteriori error estimator for the augmented formulation. The proof of reliability makes use of a global inf–sup condition, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart–Thomas operator. In turn, for the efficiency of the estimator we use inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works.

The rest of this paper is organized as follows. In Section 2 we present the main aspects of the continuous problem, which includes the geometry and the coupled model. Then, in Section 3 we introduce and analyze the fully–mixed variational formulation. More precisely, we show the unique solvability of the continuous scheme and derive suitable hypotheses on the discrete subspaces ensuring that the associated Galerkin scheme becomes well-posed. In addition, we provide concrete examples of finite element spaces in 2D and 3D satisfying the corresponding hypotheses on the discrete subspaces. Next, in Section 4 we deal with the augmented mixed approach. We analyze the existence and uniqueness of solution of the continuous formulation, and derive suitable hypotheses on the discrete subspaces, less demanding than those introduced in Section 3, ensuring the well-posedness of the associated Galerkin scheme. Then we provide suitable choices of finite element spaces in 2D and 3D for the augmented mixed formulation. In Section 5 we derive the residual-based a posteriori error estimator for the aforementioned scheme in 2D, and prove its reliability and efficiency. Finally, several numerical results illustrating the good performance of the methods, confirming the properties of the estimator, and showing the capability of the associated adaptive algorithm to localize the singularities of the solution, are reported in Section 6.

We end this section with some notations to be used below. In what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^n, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^n.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\ \mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see, e.g. [12]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. On the other hand, the following symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda\rangle_\Gamma := \int_\Gamma \xi\,\lambda \quad \forall\,\xi,\,\lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda}\rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall\,\boldsymbol{\xi},\,\boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^n$, $P_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. The vector and tensor versions of $P_k(S)$, denoted by $\mathbf{P}_k(S)$ and $\mathbb{P}_k(S)$, respectively, which are defined component-wise by $P_k(S)$, might also be required. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The coupled problem

In order to describe the geometry of the problem, we let Ω_{S} and Ω_{D} be bounded and simply connected polyhedral domains in $\mathbb{R}^n$, $n \in \{2,3\}$, such that $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}}\backslash\bar{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}}\backslash\bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Figure 2.1 below) and by $\{\mathbf{t}_1, \mathbf{t}_2\}$, when $n = 3$.

The model consists of two separate groups of equations and a set of coupling terms. In Ω_{S}, the governing equations are those of the Stokes problem, which are written in the following velocity-pressure-stress formulation:

$$\begin{aligned} \boldsymbol{\sigma}_{\mathrm{S}} = -p_{\mathrm{S}}\,\mathrm{I} + 2\,\nu\,\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}, \\ \mathrm{div}\ \mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}, \end{aligned} \tag{2.1}$$

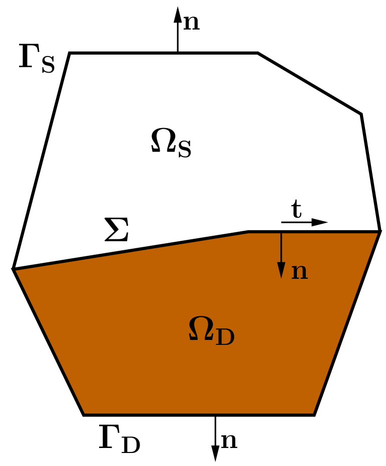

Figure 2.1: Sketch of a 2D geometry where our Stokes–Darcy model is considered.

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_S$ is the fluid velocity, p_S is the pressure, $\boldsymbol{\sigma}_S$ is the stress tensor, I is the $n \times n$ identity matrix, $\mathbf{f}_S$ is a known source term, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and

$$\mathbf{e}(\mathbf{u}_S) := \frac{1}{2}\left(\nabla \mathbf{u}_S + (\nabla \mathbf{u}_S)^{\mathtt{t}}\right)$$

is the strain tensor (or symmetric part of the velocity gradient). Now, introducing the vorticity (or skew–symmetric part of the velocity gradient) $\boldsymbol{\gamma}_S = \frac{1}{2}(\nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^{\mathtt{t}})$ as a further unknown, and using that $\operatorname{tr}(\nabla \mathbf{u}_S) = \operatorname{div} \mathbf{u}_S = 0$ in Ω_S, and the relation $\nabla \mathbf{u}_S - \boldsymbol{\gamma}_S = \mathbf{e}(\mathbf{u}_S)$ in Ω_S, we observe that the equations in (2.1) can be rewritten equivalently as

$$\begin{gathered}
\frac{1}{2\nu}\boldsymbol{\sigma}_S^{\mathtt{d}} = \nabla \mathbf{u}_S - \boldsymbol{\gamma}_S \quad \text{in} \quad \Omega_S, \qquad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S, \\
\boldsymbol{\sigma}_S = \boldsymbol{\sigma}_S^{\mathtt{t}} \quad \text{in} \quad \Omega_S, \qquad p_S = -\frac{1}{n}\operatorname{tr}\boldsymbol{\sigma}_S \quad \text{in} \quad \Omega_S, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S,
\end{gathered} \tag{2.2}$$

where tr stands for the usual trace of tensors, that is, $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}$, and

$$\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\mathrm{I},$$

is the deviatoric part of tensor $\boldsymbol{\tau}$. On the other hand, in Ω_D we consider the following Darcy model:

$$\begin{gathered}
\mathbf{u}_D = -\mathrm{K}\,\nabla p_D \quad \text{in} \quad \Omega_D, \qquad \operatorname{div}\mathbf{u}_D = f_D \quad \text{in} \quad \Omega_D, \\
\mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_D,
\end{gathered} \tag{2.3}$$

where $\mathbf{u}_D$ and p_D denote the velocity and pressure, respectively, and the source term f_D is such that $\int_{\Omega_D} f_D = 0$. The matrix valued function K, describing the permeability of Ω_D divided by the viscosity ν, satisfies $\mathrm{K}^{\mathtt{t}} = \mathrm{K}$, and has $L^\infty(\Omega_D)$ components. Also, we assume that there exists $C_K > 0$ such that

$$\mathbf{w} \cdot \mathrm{K}(x)\,\mathbf{w} \geq C_K \|\mathbf{w}\|^2, \tag{2.4}$$

for almost all $x \in \Omega_D$, and for all $\mathbf{w} \in \mathbb{R}^n$.

Finally, the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_\mathrm{S}\cdot\mathbf{n} \,&=\, \mathbf{u}_\mathrm{D}\cdot\mathbf{n} \qquad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_\mathrm{S}\,\mathbf{n} + \sum_{l=1}^{n-1}\pi_l^{-1}\,(\mathbf{u}_\mathrm{S}\cdot\mathbf{t}_l)\,\mathbf{t}_l \,&=\, -\,p_\mathrm{D}\,\mathbf{n} \qquad \text{on} \quad \Sigma\,, \end{aligned} \tag{2.5}$$

where $\{\pi_1, ..., \pi_{n-1}\}$ is a set of positive frictional constants, which are determined experimentally.

3 The fully-mixed approach

The purpose of this section is to generalize the results provided in [35], introducing and analyzing a new fully-mixed variational formulation, together with its corresponding Galerkin scheme, for the coupled system given by the set of equations (2.2), (2.3) and (2.5). As already remarked in Section 1, the main novelty with respect to the approach in [35] is the utilization now of $\mathbf{e}(\mathbf{u}_\mathrm{S})$ instead of $\nabla\mathbf{u}_\mathrm{S}$ in the definition of the stress tensor $\boldsymbol{\sigma}_\mathrm{S}$ (cf. (2.1)). We study the well-posedness of both, the continuous and discrete problems, and introduce feasible choices of finite element spaces for the 2D and 3D cases.

3.1 The continuous formulation

In this section, we proceed analogously to [35] and introduce a mixed formulation for the coupled problem. To do this, let us first introduce further notations and definitions. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, we denote

$$(u,v)_\star \,:=\, \int_{\Omega_\star} u\,v, \qquad (\mathbf{u},\mathbf{v})_\star \,:=\, \int_{\Omega_\star}\mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star \,:=\, \int_{\Omega_\star}\boldsymbol{\sigma}:\boldsymbol{\tau}\,,$$

where $\boldsymbol{\sigma}:\boldsymbol{\tau} \,=\, \mathrm{tr}\,(\boldsymbol{\sigma}^\mathrm{t}\boldsymbol{\tau}) \,=\, \sum_{i,j=1}^{n}\sigma_{ij}\tau_{ij}$. In addition, we let $\mathbb{L}^2_\mathrm{skew}(\Omega_\mathrm{S})$ be the subspace of skew–symmetric tensors of $\mathbb{L}^2(\Omega_\mathrm{S})$, that is

$$\mathbb{L}^2_\mathrm{skew}(\Omega_\mathrm{S}) := \left\{\boldsymbol{\eta}\in\mathbb{L}^2(\Omega_\mathrm{S}) : \boldsymbol{\eta}+\boldsymbol{\eta}^\mathrm{t} = \mathbf{0}\right\}.$$

Furthermore, we need to introduce the space

$$\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D}) \,:=\, \{\mathbf{v}\,\in\mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) : \quad \mathbf{v}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_\mathrm{D}\}\,,$$

and the space of traces $\mathbf{H}^{1/2}_{00}(\Sigma) \,:=\, [\,H^{1/2}_{00}(\Sigma)]^n$, where

$$H^{1/2}_{00}(\Sigma) \,:=\, \left\{ v|_\Sigma : \quad v\,\in\,H^1(\Omega_\mathrm{S})\,, \quad v\,=\,0 \quad \text{on} \quad \Gamma_\mathrm{S} \right\}.$$

Observe that, if $E_{0,\mathrm{S}} : H^{1/2}(\Sigma)\to L^2(\partial\Omega_\mathrm{S})$, is the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) \,:=\, \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_\mathrm{S} \end{cases} \qquad \forall\,\psi\,\in\,H^{1/2}(\Sigma)\,,$$

then, the space $H^{1/2}_{00}(\Sigma)$ can be defined equivalently as

$$H^{1/2}_{00}(\Sigma) \,=\, \left\{\psi\,\in\,H^{1/2}(\Sigma) : \quad E_{0,\mathrm{S}}(\psi)\,\in\,H^{1/2}(\partial\Omega_\mathrm{S})\right\},$$

endowed with the norm $\|\psi\|_{1/2,00,\Sigma} := \|E_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_{\mathrm{S}}}$. The dual space of $\mathbf{H}^{1/2}_{00}(\Sigma)$ is denoted by $\mathbf{H}^{-1/2}_{00}(\Sigma)$.

Now, to proceed with the derivation of our mixed problem, let us now define two additional unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1/2}_{00}(\Sigma), \quad \text{and} \quad \lambda := p_{\mathrm{D}} \in H^{1/2}(\Sigma). \tag{3.1}$$

Notice that, in principle, the spaces for $\mathbf{u}_{\mathrm{S}}$ and p_{D} do not allow enough regularity for the traces $\boldsymbol{\varphi}$ and λ to exist. However, solutions of (2.2) and (2.3) have these unknowns in $\mathbf{H}^1(\Omega_{\mathrm{S}})$ and $H^1(\Omega_{\mathrm{D}})$, respectively.

In this way, to derive the weak formulation of the coupled system (2.2)–(2.3)–(2.5), we test the first equations of (2.2) and (2.3) with arbitrary $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$, respectively, integrate by parts, utilize the identity $\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}} : \boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}} : \boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}}$, and impose weakly the remaining equations, to obtain the variational problem: Find $(\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda) \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma)$ and $(\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}) \in \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L^2(\Omega_{\mathrm{D}})$, such that:

$$\begin{aligned}
\frac{1}{2\nu}\,(\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}}, \boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}\rangle_{\Sigma} + (\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \;&=\; 0\\
(\mathrm{K}^{-1}\,\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (\mathrm{div}\;\mathbf{v}_{\mathrm{D}}, p_{\mathrm{D}})_{\mathrm{D}} - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda\rangle_{\Sigma} \;&=\; 0\\
(\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \;&=\; -\,(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\\
(\mathrm{div}\,\mathbf{u}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} \;&=\; (f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}\\
(\boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} \;&=\; 0\\
\langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_{\Sigma} + \langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_{\Sigma} \;&=\; 0\\
\langle \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_{\Sigma} - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} \;&=\; 0,
\end{aligned} \tag{3.2}$$

for all $(\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi) \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma)$ and $(\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L^2(\Omega_{\mathrm{D}})$, where

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} = \sum_{l=1}^{n-1} \pi_l^{-1}\,\langle \boldsymbol{\varphi}\cdot\mathbf{t}_l, \boldsymbol{\psi}\cdot\mathbf{t}_l\rangle_{\Sigma}\,. \tag{3.3}$$

Observe that the symmetry of $\boldsymbol{\sigma}_{\mathrm{S}}$ is imposed weakly by the fifth equation in (3.2).

Next, analogously to the proof of [35, Lemma 3.5], it is easy to see that (3.2) has a one dimensional kernel $\{(-\mathrm{I}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 1)\,,\,(\mathbf{0}, 1)\}$. Then, we avoid the non-uniqueness of (3.2) by requiring from now on that $p_{\mathrm{D}} \in L^2_0(\Omega_{\mathrm{D}})$, where

$$L^2_0(\Omega_{\mathrm{D}}) := \left\{ q \in L^2(\Omega_{\mathrm{D}}) : \quad \int_{\Omega_{\mathrm{D}}} q = 0 \right\}.$$

On the other hand, for convenience of the subsequent analysis, we consider the decomposition

$$\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) = \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \oplus P_0(\Omega_{\mathrm{S}})\mathrm{I}\,, \tag{3.4}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) := \left\{ \boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \;:\; \int_{\Omega_{\mathrm{S}}} \mathrm{tr}\;\boldsymbol{\sigma} = 0 \right\},$$

and redefine the stress tensor as $\boldsymbol{\sigma}_{\mathrm{S}} = \boldsymbol{\sigma}_{\mathrm{S}} + \mu\,\mathrm{I}$, with the new unknowns $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mu \in \mathbb{R}$.

In this way, the first and last equations of (3.2) are rewritten, equivalently as

$$\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} + (\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}\rangle_{\Sigma} + (\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = 0 \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}), \tag{3.5}$$

$$\rho\,\langle \boldsymbol{\varphi}\cdot\mathbf{n}, 1\rangle_{\Sigma} = 0 \qquad \forall\, \rho \in \mathbb{R}, \tag{3.6}$$

$$\langle \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_{\Sigma} - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \mu\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_{\Sigma} = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma). \tag{3.7}$$

Now, it is quite clear that there are many different ways of ordering the variational system described above, but in this section we proceed as in [35, Section 2.3], and adopt one leading to a doubly-mixed structure (also known as twofold saddle point operator equation). To this end, we group spaces, unknowns, and test functions as follows:

$$\begin{gathered}
\mathbb{X}_0 := \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma),\\
\mathbb{M}_0 := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L^2_0(\Omega_{\mathrm{D}}) \times \mathbb{R},\\
\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda) \in \mathbb{X}_0, \quad \underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}, \mu) \in \mathbb{M}_0,\\
\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_0, \quad \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \rho) \in \mathbb{M}_0,
\end{gathered} \tag{3.8}$$

where $\mathbb{X}_0$ and $\mathbb{M}_0$ are respectively endowed with the norms

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}} + \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} + \|\xi\|_{1/2,\Sigma}$$

and

$$\|\underline{\mathbf{v}}\|_{\mathbb{M}} := \|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |\rho|.$$

Here, $\mathbb{X}$ and $\mathbb{M}$ denote the product spaces defined respectively as $\mathbb{X}_0$ and $\mathbb{M}_0$, but considering the spaces $\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$ and $L^2(\Omega_{\mathrm{D}})$, instead of $\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ and $L^2_0(\Omega_{\mathrm{D}})$. Hence, the variational system (3.2) with the new equations (3.5) – (3.7), reads: Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ such that

$$\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_0,\\
\mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \rho) \in \mathbb{M}_0,
\end{aligned} \tag{3.9}$$

where

$$\mathcal{F}(\underline{\boldsymbol{\tau}}) := 0, \qquad \mathcal{G}(\underline{\mathbf{v}}) = \mathcal{G}((\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \rho)) := -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (f_{\mathrm{D}}, q_{\mathrm{D}}), \tag{3.10}$$

and $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) := {}& a((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) + b((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda))\\
&+ b((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi)) - c((\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda), (\boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi)),
\end{aligned} \tag{3.11}$$

with

$$\begin{aligned}
a((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) &:= \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} + (\mathrm{K}^{-1}\,\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}},\\
b((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi)) &:= (\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_{\Sigma},\\
c((\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda), (\boldsymbol{\eta}_{\mathrm{S}}, \boldsymbol{\psi}, \xi)) &:= \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_{\Sigma} - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_{\Sigma},
\end{aligned} \tag{3.12}$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} + \rho\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_{\Sigma}. \tag{3.13}$$

3.2 Analysis of the continuous formulation

In this section we proceed similarly as in [35, Section 3] and use the classical Babuška-Brezzi theory to show that (3.9) is well-posed. To this end, we first collect some known results that will serve for the forthcoming analysis. We begin by recalling that the following inequalities hold

$$\|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}} \,\geq\, C_{\mathrm{D}}\,\|\mathbf{v}_{\mathrm{D}}\|^2_{\mathrm{div}\,,\Omega_{\mathrm{D}}} \qquad \forall\,\mathbf{v}_{\mathrm{D}}\,\in\,\mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}}) \quad \text{such that} \quad \mathrm{div}\,\mathbf{v}_{\mathrm{D}}\,\in\,P_0(\Omega_{\mathrm{D}})\,, \tag{3.14}$$

and

$$C_{\mathrm{S}}\,\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \leq \|\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \quad \forall\,\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}). \tag{3.15}$$

For (3.14) we refer to [35, Lemma 3.2], whereas (3.15) is proved in [2, Lemma 3.1] (see also [12, Chapter IV]).

The following lemma will be employed in what follows. For its proof we refer to [35, Lemma 3.4] (see also [29, Lemma 2.1] for a nonlinear version of it).

Lemma 3.1 *Let $(X,\langle\cdot,\cdot\rangle_X)$ and $(Y,\langle\cdot,\cdot\rangle_Y)$ be Hilbert spaces. Let $a : X\times X \to \mathbb{R}$, $b : X\times Y \to \mathbb{R}$, and $c : Y\times Y \to \mathbb{R}$ be bounded bilinear forms, and let $A : (X\times Y)\times(X\times Y) \to \mathbb{R}$ be the global bilinear form defined by*

$$A((x,y),(z,w)) := a(x,z) + b(z,y) + b(x,w) - c(y,w) \qquad \forall\,(x,y),\,(z,w) \in X\times Y\,.$$

Assume that

i) *there exists $\bar{\alpha} > 0$ such that $a(x,x) \,\geq\, \bar{\alpha}\,\|x\|^2_X \quad \forall\,x \in X$,*

ii) *there exists $\bar{\beta} > 0$ such that $\displaystyle\sup_{x\in X\backslash 0} \frac{b(x,y)}{\|x\|_X} \,\geq\, \bar{\beta}\,\|y\|_Y \quad \forall\,y \in Y$,*

iii) *$c(y,y) \,\geq\, 0 \quad \forall\,y \in Y$.*

Then, the linear operator induced by A, namely $\mathbb{A} : X\times Y \to X\times Y$ defined by

$$\langle\mathbb{A}(u,v),(z,w)\rangle_{X\times Y} = A((u,v),(z,w)) \qquad \forall\;(u,v),(z,w) \in X\times Y,$$

is invertible.

In the sequel, for the sake of simplicity, whenever a bilinear form A induces an invertible operator, we will simply say that the bilinear form A is invertible.

We begin the analysis of (3.9) by proving the inf-sup condition associated to $\mathcal{B}$.

Lemma 3.2 *There exists $\beta > 0$ such that*

$$\sup_{\underline{\boldsymbol{\tau}}\,\in\,\mathbb{X}_0\backslash\mathbf{0}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}}} \,\geq\, \beta\,\|\underline{\mathbf{v}}\|_{\mathbb{M}} \qquad \forall\,\underline{\mathbf{v}}\,\in\,\mathbb{M}_0\,. \tag{3.16}$$

Proof. Analogously to the proof of [35, Lemma 3.6], we observe that the diagonal character of $\mathcal{B}$ (cf. (3.13)) guarantees that (3.16) is equivalent to the following three independent inf-sup conditions:

$$\sup_{\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\backslash \mathbf{0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \beta_{\mathrm{S}}\,\|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \qquad \forall\,\mathbf{v}_{\mathrm{S}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{S}}), \tag{3.17}$$

$$\sup_{\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\backslash \mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \;\geq\; \beta_{\mathrm{D}}\,\|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\,q_{\mathrm{D}} \,\in\, L^2_0(\Omega_{\mathrm{D}})\,, \tag{3.18}$$

$$\sup_{\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)\backslash \mathbf{0}} \frac{\rho\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00,\Sigma}} \;\geq\; \beta_\Sigma\,|\rho| \qquad \forall\,\rho \,\in\, \mathbb{R}\,, \tag{3.19}$$

with β_{S}, β_{D}, $\beta_\Sigma \,>\, 0$. First, given $q_{\mathrm{D}} \,\in\, L^2_0(\Omega_{\mathrm{D}})$, we define $\mathbf{v}_{\mathrm{D}} \,:=\, \nabla z$, where $z \,\in\, H^1_\Sigma(\Omega_{\mathrm{D}})$ is the unique solution of the boundary value problem:

$$\Delta z \,=\, q_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad z \,=\, 0 \quad \text{on} \quad \Sigma, \quad \frac{\partial z}{\partial \mathbf{n}} \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}.$$

It follows that $\mathbf{v}_{\mathrm{D}} \,\in\, \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ and $\mathrm{div}\,\mathbf{v}_{\mathrm{D}} \,=\, q_{\mathrm{D}}$, which yields the surjectivity of the operator $\mathrm{div} \,:\, \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \to L^2_0(\Omega_{\mathrm{D}})$, which is (3.18). With similar arguments one can prove that $\mathbf{div} \,:\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \to \mathbf{L}^2(\Omega_{\mathrm{S}})$ is also surjective, which is (3.17). Finally, we recall that the proof of the inf-sup condition (3.19) relies on the existence of a fixed element $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$ such that $\langle \boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$. For the construction of such function $\boldsymbol{\psi}_0$ we simply refer to [35, Section 3.2] or [37, Section 3.2]. □

Now, let $\mathbb{V}$ be the kernel of $\mathcal{B}$, that is

$$\mathbb{V} \,:=\, \big\{\, \underline{\boldsymbol{\tau}} \,\in\, \mathbb{X}_0 : \quad \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \,=\, 0 \quad \forall\,\underline{\mathbf{v}} \,\in\, \mathbb{M}_0 \,\big\}\,.$$

From the definition of $\mathcal{B}$ (cf. (3.13)), it is easy to see that $\mathbb{V} \,=\, \mathbb{V}_1 \,\times\, \mathbb{V}_2$, where

$$\mathbb{V}_1 \,=\, \tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \quad \text{and} \quad \mathbb{V}_2 \,=\, \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma)\,,$$

with

$$\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}}) \,:=\, \Big\{\, \boldsymbol{\tau}_{\mathrm{S}} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) : \quad \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}} \,=\, \mathbf{0} \,\Big\}\,,$$

$$\tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \,:=\, \Big\{\, \mathbf{v}_{\mathrm{D}} \,\in\, \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \mathrm{div}\,\mathbf{v}_{\mathrm{D}} \,\in\, \mathbb{P}_0(\Omega_{\mathrm{D}}) \,\Big\}\,,$$

and

$$\tilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \,:=\, \Big\{\, \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma \,=\, 0 \,\Big\}\,.$$

The following lemma establishes the invertibility of $\mathcal{A}$ on $\mathbb{V}$.

Lemma 3.3 *The bilinear form $\mathcal{A}$ is invertible on $\mathbb{V}\times\mathbb{V}$.*

Proof. Due to the structure of $\mathcal{A}$, in what follows we apply Lemma 3.1, that is, we verify that the bilinear forms a, b and c (cf. (3.12)) satisfy the corresponding hypotheses i), ii) and iii) on $\mathbb{V} = \mathbb{V}_1\times\mathbb{V}_2$. First, according to the definition of $\mathbb{V}_1$, and utilizing inequalities (3.14) and (3.15), it is not difficult to see that a satisfies i) (see [35, Lemma 3.7] for details). Next, due to the diagonal character of b, it is easy to see that b verifies ii) on $\mathbb{V}_1 \times \mathbb{V}_2$ if and only if there exist $\beta^{\mathrm{S}}_\Sigma$, $\beta^{\mathrm{D}}_\Sigma \,>\, 0$ such that

$$\sup_{\boldsymbol{\tau}_{\mathrm{S}} \in \tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})\backslash \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma \,+\, (\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \beta^{\mathrm{S}}_\Sigma \Big\{ \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \,+\, \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \Big\} \tag{3.20}$$

$\forall\,(\boldsymbol{\eta}_{\mathrm{S}},\boldsymbol{\psi})\,\in\,\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})\times\tilde{\mathbf{H}}^{1/2}_{00}(\Sigma)$, and

$$\sup_{\mathbf{v}_{\mathrm{D}}\,\in\,\tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\backslash\mathbf{0}}\;\frac{\langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma}}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}}\;\geq\;\beta^{\mathrm{D}}_{\Sigma}\,\|\xi\|_{1/2,\Sigma}\qquad\forall\,\xi\,\in\,H^{1/2}(\Sigma)\,. \tag{3.21}$$

Then we observe that (3.20) follows from a slight modification of [32, Lemma 4.3]. In addition, using similar arguments utilized in [37, Lemma 3.3] one can obtain that the operator $\mathbf{v}\;\rightarrow\;\mathbf{v}\cdot\mathbf{n}$ from $\tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ to $H^{-1/2}(\Sigma)$ is surjective, which yields (3.21). Finally, it is quite straightforward from (3.3) and the definition of c (cf. (3.12)), that for each $(\boldsymbol{\eta}_{\mathrm{S}},\boldsymbol{\psi},\xi)\in\mathbb{V}_2$ there holds

$$c((\boldsymbol{\eta}_{\mathrm{S}},\boldsymbol{\psi},\xi),(\boldsymbol{\eta}_{\mathrm{S}},\boldsymbol{\psi},\xi))=\sum_{l=1}^{n-1}\pi_l^{-1}\|\boldsymbol{\psi}\cdot\mathbf{t}_l\|^2_{0,\Sigma}\geq 0, \tag{3.22}$$

which shows that c verifies hypothesis iii), and the proof is concluded. $\square$

We are now in position of establishing the main results of this section.

Theorem 3.4 *For each pair* $(\mathcal{F},\mathcal{G})\,\in\,\mathbb{X}_0'\times\mathbb{M}_0'$ *there exists a unique* $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\,\in\,\mathbb{X}_0\times\mathbb{M}_0$ *solution to* (3.9)*. In addition, there exists a constant* $C>0$*, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\|_{\mathbb{X}\times\mathbb{M}}\;\leq\;C\left\{\,\|\mathcal{F}\|_{\mathbb{X}_0'}\,+\,\|\mathcal{G}\|_{\mathbb{M}_0'}\,\right\}.$$

Proof. It follows from Lemmata 3.2 and 3.3, and a straightforward application of the classical Babuška-Brezzi theory. $\square$

Theorem 3.5 *Let* $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\,\in\,\mathbb{X}_0\times\mathbb{M}_0$ *be the unique solution of the variational formulation* (3.9) *with* $\mathcal{F}$ *and* $\mathcal{G}$ *given by* (3.10)*, and define* $p_{\mathrm{S}}:=-\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S}})$*. Then,* $\mathbf{u}_{\mathrm{S}}\in\mathbf{H}^1(\Omega_{\mathrm{S}})$*,* $p_{\mathrm{D}}\in H^1(\Omega_{\mathrm{D}})$*,* $\boldsymbol{\varphi}=-\mathbf{u}_{\mathrm{S}}$ *on* Σ*,* $\lambda=p_{\mathrm{D}}$ *on* Σ*, and we have a solution of the system* (2.2)*,* (2.3) *and* (2.5)*.*

Proof. It basically follows by applying integration by parts backwardly in (3.9), and using suitable test functions. We omit further details. $\square$

3.3 Galerkin scheme of the fully-mixed approach

In this section we introduce the Galerkin scheme of problem (3.9) and analyze its well-posedness by establishing suitable assumptions on the discrete subspaces involved. We begin by selecting a set of arbitrary discrete spaces, namely

$$\begin{gathered}\mathbf{H}_h(\Omega_\star)\;\subseteq\;\mathbf{H}(\mathrm{div}\,;\Omega_\star)\,,\quad L_h(\Omega_\star)\;\subseteq\;L^2(\Omega_\star)\,,\quad\star\in\{\mathrm{S},\mathrm{D}\},\\ \Lambda^{\mathrm{S}}_h(\Sigma)\;\subseteq\;H^{1/2}_{00}(\Sigma)\,,\quad\Lambda^{\mathrm{D}}_h(\Sigma)\;\subseteq\;H^{1/2}(\Sigma)\,,\quad\mathbb{S}_h(\Omega_{\mathrm{S}})\subseteq\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})\,.\end{gathered} \tag{3.23}$$

Then, we define the subspaces:

$$\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \,:\, \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}), \quad \forall\, \mathbf{c} \in \mathbb{R}^n\}\,,\\
\mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) &:= \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\,,\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \big\{\mathbf{v}_h \in \mathbf{H}_h(\Omega_{\mathrm{D}}) : \quad \mathbf{v}_h\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\big\},\\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= [L_h(\Omega_{\mathrm{S}})]^n,\\
L_{h,0}(\Omega_{\mathrm{D}}) &:= L_h(\Omega_{\mathrm{D}}) \cap L^2_0(\Omega_{\mathrm{D}})\,,\\
\boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) &:= [\Lambda^{\mathrm{S}}_h(\Sigma)]^n.
\end{aligned} \tag{3.24}$$

In this way, grouping global discrete subspaces and corresponding unknowns as follows:

$$\begin{gathered}
\mathbb{X}_{h,0} := \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) \times \Lambda^{\mathrm{D}}_h(\Sigma)\,,\\
\mathbb{M}_{h,0} := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}}) \times \mathbb{R}\,,
\end{gathered} \tag{3.25}$$

$$\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\gamma}_{\mathrm{S},h}, \boldsymbol{\varphi}_h, \lambda_h) \,\in\, \mathbb{X}_{h,0}\,, \quad \underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h}, \mu_h) \,\in\, \mathbb{M}_{h,0}\,,$$

we find that the discrete version of problem (3.9) reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ such that

$$\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) \,+\, \mathcal{B}(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{u}}_h) &= \mathcal{F}(\underline{\boldsymbol{\tau}}_h) \qquad \forall\, \underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h, \xi_h) \,\in\, \mathbb{X}_{h,0}\,,\\
\mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}_h) &= \mathcal{G}(\underline{\mathbf{v}}_h) \qquad \forall\, \underline{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{S},h}, q_{\mathrm{D},h}, \rho_h) \,\in\, \mathbb{M}_{h,0}\,.
\end{aligned} \tag{3.26}$$

Next, we proceed analogously to [35, Section 4] and establish general hypotheses on the finite element subspaces (3.23) and (3.24), ensuring the well-posedness of (3.26). We begin by noticing that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$ and $L_{h,0}(\Omega_{\mathrm{D}})$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h(\Omega_{\mathrm{S}})$ and constant polynomials from $L_h(\Omega_{\mathrm{D}})$. This request is certainly satisfied if we assume:

(H.0) $[P_0(\Omega_{\mathrm{S}})]^{n\times n} \subseteq \mathbb{H}_h(\Omega_{\mathrm{S}})$ *and* $P_0(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$.

In particular, it follows that $\mathrm{I} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$ for all h, and hence there holds:

$$\mathbb{H}_h(\Omega_{\mathrm{S}}) \,=\, \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \,\oplus\, P_0(\Omega_{\mathrm{S}})\,\mathrm{I}\,. \tag{3.27}$$

Now, using the same diagonal argument utilized in the proof of Lemma 3.2, we observe that the discrete inf-sup condition $\mathcal{B}$ holds if we assume:

(H.1) *There exist* $\tilde{\beta}_{\mathrm{S}}\,,\ \tilde{\beta}_{\mathrm{D}} \,>\, 0$*, independent of* h*, and there exists* $\boldsymbol{\psi}_0 \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma)$*, such that*

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{h,0}(\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \,\geq\, \tilde{\beta}_{\mathrm{S}}\,\|\mathbf{v}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} \qquad \forall\, \mathbf{v}_{\mathrm{S},h} \,\in\, \mathbf{L}_h(\Omega_{\mathrm{S}}), \tag{3.28}$$

$$\sup_{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\backslash\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_{\mathrm{D},h}, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \,\geq\, \tilde{\beta}_{\mathrm{D}}\,\|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\, q_{\mathrm{D},h} \,\in\, L_{h,0}(\Omega_{\mathrm{D}})\,, \tag{3.29}$$

$$\boldsymbol{\psi}_0 \,\in\, \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \quad \forall\, h \quad \textit{and} \quad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \neq 0\,. \tag{3.30}$$

In particular, note that (3.30) implies the inf-sup condition

$$\sup_{\boldsymbol{\psi}_h \,\in\, \boldsymbol{\Lambda}_h(\Sigma)\backslash \mathbf{0}} \frac{\rho_h\,\langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1\rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma}} \;\geq\; \tilde{\beta}_\Sigma\,|\rho_h| \qquad \forall\, \rho_h \,\in\, \mathbb{R}\,. \tag{3.31}$$

We now look at the discrete kernel of $\mathcal{B}$, which is defined by

$$\mathbb{V}_h \;:=\; \big\{\, \underline{\boldsymbol{\tau}}_h \,\in\, \mathbb{X}_{h,0} : \quad \mathcal{B}(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h) \;=\; 0 \quad \forall\, \underline{\mathbf{v}}_h \,\in\, \mathbb{M}_{h,0} \,\big\}\,.$$

In order to have a more explicit definition of $\mathbb{V}_h$, we introduce the following assumption:

(H.2) $\mathbf{div}\,\mathbb{H}_h(\Omega_{\mathrm{S}}) \,\subseteq\, \mathbf{L}_h(\Omega_{\mathrm{S}})$ *and* $\mathrm{div}\,\mathbf{H}_h(\Omega_{\mathrm{D}}) \,\subseteq\, L_h(\Omega_{\mathrm{D}})$.

It follows from **(H.2)** and the definition of $\mathcal{B}$ (cf. (3.13)) that $\mathbb{V}_h = \mathbb{V}_{1,h} \times \mathbb{V}_{2,h}$, where

$$\mathbb{V}_{1,h} \;=\; \tilde{\mathbb{H}}_{h,0}(\Omega_{\mathrm{S}}) \times \tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \quad \text{and} \quad \mathbb{V}_{2,h} \;=\; \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma)\,,$$

with

$$\tilde{\mathbb{H}}_{h,0}(\Omega_{\mathrm{S}}) \;:=\; \Big\{\, \boldsymbol{\tau}_h \,\in\, \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) : \, \mathbf{div}\,\boldsymbol{\tau}_h \;=\; \mathbf{0} \,\Big\},$$

$$\tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \;:=\; \Big\{\, \mathbf{v}_h \,\in\, \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) : \quad \mathrm{div}\,\mathbf{v}_h \,\in\, P_0(\Omega_{\mathrm{D}}) \,\Big\}\,,$$

and

$$\tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma) \;:=\; \Big\{ \boldsymbol{\psi}_h \,\in\, \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) : \quad \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1\rangle_\Sigma \;=\; 0 \,\Big\}\,.$$

In addition, regarding the inf-sup condition of b on $\mathbb{V}_h$, we also define the subspace

$$\tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}}) \;:=\; \Big\{\, \boldsymbol{\tau}_h \,\in\, \mathbb{H}_h(\Omega_{\mathrm{S}}) : \, \mathbf{div}\,\boldsymbol{\tau}_h \;=\; \mathbf{0} \,\Big\}. \tag{3.32}$$

Then, applying the same diagonal argument employed in the proof of Lemma 3.3, we deduce that b satisfies the discrete inf-sup condition on $\mathbb{V}_h$ if and only if the following hypothesis holds:

(H.3) *There exist positive constants* $\tilde{\beta}_\Sigma^{\mathrm{S}}$, $\tilde{\beta}_\Sigma^{\mathrm{D}}$ *, independent of* h*, such that*

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \,\in\, \tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})\backslash \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma \,+\, (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \tilde{\beta}_\Sigma^{\mathrm{S}} \Big\{ \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \,+\, \|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} \Big\}\,, \tag{3.33}$$

for all $(\boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h) \,\in\, \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma)$*, and*

$$\sup_{\mathbf{v}_{\mathrm{D},h} \,\in\, \tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\backslash \mathbf{0}} \frac{\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \;\geq\; \tilde{\beta}_\Sigma^{\mathrm{D}}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h^{\mathrm{D}}(\Sigma)\,. \tag{3.34}$$

In particular, given $(\boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h) \in \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma)$, we observe that (3.33), and the fact that $\langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1\rangle_\Sigma = 0$, imply

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \tilde{\mathbb{H}}_{h,0}(\Omega_{\mathrm{S}})\backslash \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma \,+\, (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \tilde{C}\left\{\|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \,+\, \|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\right\}, \tag{3.35}$$

which corresponds to the discrete version of (3.20).

The following theorem establishes the well-posedness of problem (3.26) and the corresponding Céa estimate.

Theorem 3.6 *Assume that hypotheses* **(H.0)**, **(H.1)**, **(H.2)**, *and* **(H.3)** *hold. Then, the Galerkin scheme* (3.26) *has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$, *and there exists* $C_1 > 0$, *independent of* h, *such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \;\leq\; C_1 \left\{ \|\mathcal{F}|_{\mathbb{X}_{h,0}}\|_{\mathbb{X}_{h,0}'} \,+\, \|\mathcal{G}|_{\mathbb{M}_{h,0}}\|_{\mathbb{M}_{h,0}'} \right\}.$$

In addition, there exists $C_2 > 0$, *independent of* h, *such that*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \,+\, \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \;\leq\; C_2 \left\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_{h,0}} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \,+\, \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_{h,0}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \right\}, \tag{3.36}$$

where $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *is the unique solution of* (3.9).

Proof. It follows by applying similar arguments to those utilized in Section 3.2. We omit further details. $\square$

3.4 Particular choices of discrete subspaces

We now specify concrete examples of finite element subspaces in 2D and 3D satisfying the hypotheses introduced in the previous section. To this end, we let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles (in $\mathbb{R}^2$) or tetrahedra (in $\mathbb{R}^3$) of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). In addition, we let b_T be the element bubble function defined as the unique polynomial in $P_{n+1}(T)$ vanishing on ∂T with $\int_T b_T = 1$, and denote by $\mathbf{x} := (x_1, ..., x_n)^{\mathbf{t}}$ a generic vector of $\mathbb{R}^n$. Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas and bubble spaces of order 0, respectively, by

$$\mathrm{RT}_0(T) \;:=\; \mathbf{P}_0(T) \oplus P_0(T)\mathbf{x},$$

and

$$\mathrm{B}_0(T) := \begin{cases} P_0(T)\left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right) & \text{in } \mathbb{R}^2, \\ \nabla \times (b_T \mathbf{P}_0(T)) & \text{in } \mathbb{R}^3. \end{cases}$$

3.4.1 PEERS + Raviart Thomas in 2D

We define the discrete subspaces in (3.23) as follows:

$$
\begin{aligned}
\mathbf{H}_h(\Omega_S) &:= \Big\{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_S): \quad \tau_h|_T \in \mathrm{RT}_0(T) \oplus \mathrm{B}_0(T) \quad \forall\, T \in \mathcal{T}_h^S \Big\}, \\
\mathbf{H}_h(\Omega_D) &:= \Big\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,;\Omega_D): \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^D \Big\}, \\
L_h(\Omega_\star) &:= \Big\{ q_h \in L^2(\Omega_\star) \,:\, q_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\} \quad \star \in \{\mathrm{S}, \mathrm{D}\}, \\
\mathbb{S}_h(\Omega_S) &:= \Big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_S) \,:\, \boldsymbol{\eta}_h \in [\mathcal{C}(\bar{\Omega}_S)]^{2\times 2} \quad \text{and} \quad \boldsymbol{\eta}_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^S \Big\}.
\end{aligned}
\tag{3.37}
$$

We remark here that $\mathbb{H}_h(\Omega_S) \times \mathbf{L}_h(\Omega_S) \times \mathbb{S}_h(\Omega_S)$, with $\mathbb{H}_h(\Omega_S)$ and $\mathbf{L}_h(\Omega_S)$ defined as in (3.24), constitutes the classical PEERS space introduced in [1] for the mixed finite element approximation of the linear elasticity problem with Dirichlet boundary condition (see, for instance [12] or [43]). In turn, $\mathbf{H}_h(\Omega_D) \times L_h(\Omega_D)$ is the Raviart-Thomas stable element of lowest order for the mixed formulation of the Poisson problem (see, for instance [12, 28]). These facts are particularly important for the rest of the analysis, since, as we will make it clear below, all the discrete inf-sup conditions that are required in the hypotheses indicated in Section 3.3, either are already available in the literature or can be derived from related results provided there. In addition, we recall from [35, 44] that the set of normal traces of $\tilde{\mathbf{H}}_{h,\Gamma_D}(\Omega_D)$ and $\tilde{\mathbb{H}}_h(\Omega_S)$ on Σ, are defined by the subspaces of $\mathbf{L}^2(\Sigma)$ and $L^2(\Sigma)$ given, respectively, by

$$
\Phi_h(\Sigma) := \Big\{ \phi_h : \Sigma \to \mathbb{R} : \quad \phi_h|_e \in P_0(e) \qquad \forall \text{ edge } e \in \Sigma_h \Big\}, \tag{3.38}
$$

$$
\boldsymbol{\Phi}_h(\Sigma) := \Phi_h(\Sigma) \times \Phi_h(\Sigma). \tag{3.39}
$$

Next, in order to introduce the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$, we follow the simplest approach suggested in [35] and [44]. To this end, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that, since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we define

$$
\Lambda_h^S(\Sigma) := \Big\{ \psi_h \in \mathcal{C}(\Sigma): \quad \psi_h|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \Big\}, \tag{3.40}
$$

$$
\Lambda_h^D(\Sigma) = \Big\{ \xi_h \in \mathcal{C}(\Sigma): \quad \xi_h|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h} \Big\}. \tag{3.41}
$$

In what follows, we verify that the discrete spaces $\mathbb{X}_{h,0}$ and $\mathbb{M}_{h,0}$, defined by the combination of (3.24), (3.25), (3.37), (3.40) and (3.41), satisfy hypotheses (**H.0**)–(**H.3**). We start by mentioning that hypotheses **(H.0)** and **(H.2)** are straightforward from the definitions in (3.37). In turn, it is well known that the discrete inf-sup conditions (3.28) and (3.29) hold (see for instance [1, Lemma 4.4] and [12, Chapter IV], respectively). In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ satisfying (3.30) follows as explained in [35, Section 2.5] or [37, Section 3.2]. These results yield assumption **(H.1)**.

Next, concerning hypothesis **(H.3)**, we will see in the sequel that sufficient conditions for (3.34) and (3.33) to hold true are the existence of positive constants C_Σ^{S}, $C_\Sigma^{\mathrm{D}} > 0$, independent of h, such that the following discrete inf-sup conditions are satisfied:

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \;\geq\; C_\Sigma^{\mathrm{D}} \, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h^{\mathrm{D}}(\Sigma)\,, \tag{3.42}$$

and

$$\sup_{\substack{\boldsymbol{\phi}_h \in \boldsymbol{\Phi}_h(\Sigma) \\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\phi}_h\|_{-1/2,00,\Sigma}} \;\geq\; C_\Sigma^{\mathrm{S}} \, \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \qquad \forall\, \boldsymbol{\psi}_h \,\in\, \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)\,. \tag{3.43}$$

We first refer to (3.34). Indeed, utilizing the same arguments provided in [35, Lemma 5.2] it can be proved, under the assumption of quasi-uniformity of the mesh around the interface Σ, that there exists a discrete stable lifting of the normal traces of $\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$, namely $\Phi_h(\Sigma)$. Then, applying [35, Lemma 4.2], it follows that the existence of such lifting, and the inf-sup condition (3.42), imply (3.34). It is important to notice that this result has been recently improved in [44, Theorem 5.1], where the assumption of quasi-uniformity of the mesh around Σ is no longer needed.

Now, regarding (3.33), analogously to the above argumentation, it follows that (3.43) is a sufficient condition for the existence of $\tilde{C}_\Sigma^{\mathrm{S}} > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \tilde{C}_\Sigma^{\mathrm{S}} \, \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \qquad \forall\, \boldsymbol{\psi}_h \,\in\, \tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma), \tag{3.44}$$

which, as we will see next, implies (3.33). In fact, given $(\boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h) \in \tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}}) \times \tilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma)$, we first employ (3.44) to derive

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma \,+\, (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \tilde{C}_\Sigma^{\mathrm{S}} \, \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \,-\, \|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \tag{3.45}$$

and then, applying [43, Theorem 4.5], we obtain

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma \,+\, (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \;\geq\; \tilde{\beta}_{\mathrm{skew}} \, \|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \tag{3.46}$$

with $\tilde{\beta}_{\mathrm{skew}} > 0$, independent of h. Then, combining these two inequalities we get (3.33), which completes the analysis of **(H.3)**.

Having verified hypotheses (**H.0**)–(**H.3**), a straightforward application of Theorem 3.6 yields the well-posedness of (3.26) and the corresponding Céa estimate.

Theorem 3.7 *Let* $\mathbb{X}_{h,0}$ *and* $\mathbb{M}_{h,0}$ *be the finite element subspaces defined by* (3.24) *and* (3.25)*, in terms of the specific discrete spaces given by* (3.37), (3.40) *and* (3.41)*. Then, the Galerkin scheme* (3.26) *has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$*, and there exist* c_1, $c_2 > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \;&\leq\; c_1 \Big\{ \|\mathcal{F}|_{\mathbb{X}_{h,0}}\|_{\mathbb{X}'_{h,0}} \,+\, \|\mathcal{G}|_{\mathbb{M}_{h,0}}\|_{\mathbb{M}'_{h,0}} \Big\}, \\ \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \,+\, \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \;&\leq\; c_2 \Big\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_{h,0}} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \,+\, \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_{h,0}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \Big\}, \end{aligned} \tag{3.47}$$

where $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *is the unique solution of* (3.9)*.*

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (3.26), under suitable regularity assumptions on the exact solution.

Theorem 3.8 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ *be the unique solutions of the continuous and discrete formulations* (3.9) *and* (3.26), *respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$, $\mathbf{div}\,\boldsymbol{\sigma}_{\rm S} \in \mathbf{H}^\delta(\Omega_{\rm S})$, $\boldsymbol{\gamma}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$, $\mathbf{u}_{\rm D} \in \mathbf{H}^\delta(\Omega_{\rm D})$, *and* $\operatorname{div}\mathbf{u}_{\rm D} \in H^\delta(\Omega_{\rm D})$. *Then,* $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$, $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, *and there exists* $C > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$
\begin{aligned}
\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|\mathbf{div}\,\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} \\
+ \|\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|\operatorname{div}\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} + & \|\boldsymbol{\gamma}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}} \Big\}.
\end{aligned}
\tag{3.48}
$$

Proof. We first recall from Theorem 3.5 that $\nabla\mathbf{u}_{\rm S} = \mathbf{e}(\mathbf{u}_{\rm S}) + \boldsymbol{\gamma}_{\rm S}$ and $\nabla p_{\rm D} = -\mathrm{K}^{-1}\mathbf{u}_{\rm D}$, which implies that $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$, $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$, whence $\boldsymbol{\varphi} = -\mathbf{u}_{\rm S}|_\Sigma \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda = p_{\rm D}|_\Sigma \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the Céa estimate (3.47), the approximation properties of the subspaces involved (see, e.g. [4], [12], [15], [28], [38], [39]), and the fact that, thanks to the trace theorems in $\Omega_{\rm S}$ and $\Omega_{\rm D}$, respectively, there holds

$$
\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \le c\|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \le c\|p_{\rm D}\|_{1+\delta,\Omega_{\rm S}}.
$$

□

3.4.2 PEERS + Raviart Thomas in 3D

Let us now define the discrete subspaces in (3.23) as follows:

$$
\begin{aligned}
\mathbf{H}_h(\Omega_{\rm S}) &:= \Big\{ \tau_h \in \mathbf{H}(\operatorname{div};\Omega_{\rm S}) : \quad \tau_h|_T \in \mathrm{RT}_0(T) \oplus \mathrm{B}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\rm S} \Big\}, \\
\mathbf{H}_h(\Omega_{\rm D}) &:= \Big\{ \mathbf{v}_h \in \mathbf{H}(\operatorname{div};\Omega_{\rm D}) : \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\rm D} \Big\}, \\
L_h(\Omega_\star) &:= \Big\{ q_h \in L^2(\Omega_\star) : q_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\} \quad \star \in \{\mathrm{S}, \mathrm{D}\}, \\
\mathbb{S}_h(\Omega_{\rm S}) &:= \Big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\rm skew}(\Omega_{\rm S}) : \boldsymbol{\eta}_h \in [\mathcal{C}(\bar\Omega_{\rm S})]^{3\times3} \quad \text{and} \quad \boldsymbol{\eta}_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^{\rm S} \Big\}.
\end{aligned}
\tag{3.49}
$$

Notice that these finite element subspaces are the 3D version of the ones defined in (3.37), considering that the vector and tensor live in $\mathbb{R}^3$ and $\mathbb{R}^{3\times3}$, respectively.

Now, in order to define the discrete spaces for the unknowns on the interface Σ, we proceed differently to Section 3.4.1 and introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}$, and define $\widehat{h}_\Sigma := \max\{\widehat{h}_K : K \in \Sigma_{\widehat{h}}\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$
\Lambda_h^{\rm S}(\Sigma) := \Big\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_K \in P_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \psi_h = 0 \quad \text{on} \quad \partial\Sigma \Big\}, \tag{3.50}
$$

$$
\Lambda_h^{\rm D}(\Sigma) = \Big\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_K \in P_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \Big\}. \tag{3.51}
$$

In this way, we define the discrete spaces $\mathbb{X}_{h,0}$ and $\mathbb{M}_{h,0}$ by combining (3.24), (3.25), (3.49), (3.50) and (3.51).

In what follows, we show that hypotheses **(H.0)**–**(H.3)** hold true. We begin by observing that the hypotheses **(H.0)** and **(H.2)** are straightforward. Also, analogously to Section 3.4.1, we notice that the inf-sup conditions (3.28) and (3.29) follow from [1, Lemma 4.4] or [12, Chapter IV]. In addition, proceeding as in [35, Section 2.5] or [37, Section 3.2] we conclude the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$ satisfying (3.30). These results yield assumption **(H.1)**.

Now, concerning the inf-sup conditions (3.33) and (3.34) in **(H.3)**, we let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$), formed by triangles of diameter h_K, and define $h_\Sigma := \max\{h_K : K \in \Sigma_h\}$. Then, defining the set of normal traces of $\tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$ and $\tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})$ as in (3.38) and (3.39) (considering triangles instead of edges), respectively, we use similar arguments utilized for the 2D case, and conclude that, on the one hand, (3.42) is sufficient condition for (3.34), and on the other hand, (3.43) is sufficient condition for (3.44), which, exactly as explained for the 2D case (cf. (3.45), (3.46)), yields (3.33). In this case, however, the 3D analogue of [44, Theorem 5.1], being an open problem, can not be employed. Therefore, in order to construct the stable discrete lifting of the normal traces of $\tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$ (respectively $\tilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})$), we need to employ some inverse inequalities on Σ, which requires quasi-uniform meshes in a neighbourhood of this interface. Furthermore, it can be proved (see e.g. the second part of the proof of [31, Lemma 7.5]) that there exists $C_0 \in (0,1)$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \le C_0 \widehat{h}_\Sigma$, the 3D version of (3.42) (respectively (3.43)) is satisfied. Note that this restriction on the meshsize explains the need of having introduced the independent partition $\Sigma_{\widehat{h}}$ of Σ. This concludes the analysis of **(H.3)**.

Having verified the hypotheses **(H.0)**–**(H.3)**, we are now in position of establishing the main results of this section. Their proofs, being basically as those of Theorems 3.7 and 3.8, are omitted.

Theorem 3.9 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *be the unique solution of* (3.9), *and let* $\mathbb{X}_{h,0}$ *and* $\mathbb{M}_{h,0}$ *be the finite element subspaces defined by* (3.24) *and* (3.25), *in terms of the specific discrete spaces given by* (3.49), (3.50) *and* (3.51). *In addition, assume that* $h_\Sigma \le C_0 \widehat{h}_\Sigma$. *Then, the Galerkin scheme* (3.26) *has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$, *and there exist* c_1, $c_2 > 0$, *independent of* h, *such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le c_1 \Big\{ \|\mathcal{F}|_{\mathbb{X}_{h,0}}\|_{\mathbb{X}'_{h,0}} + \|\mathcal{G}|_{\mathbb{M}_{h,0}}\|_{\mathbb{M}'_{h,0}} \Big\}.$$

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \le c_2 \Big\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_{h,0}} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} + \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_{h,0}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \Big\}. \tag{3.52}$$

Theorem 3.10 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ *be the unique solutions of the continuous and discrete formulations* (3.9) *and* (3.26), *respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, $\gamma_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, *and* $\operatorname{div}\mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$. *Then,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, *and there exists* $C > 0$, *independent of* h, *such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} &\le C\, h^\delta \Big\{ \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \\ &+ \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\operatorname{div}\,\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\gamma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}. \end{aligned} \tag{3.53}$$

4 The augmented mixed formulation

As mentioned before, in order to have more flexibility in the election of the discrete spaces for the Stokes domain, in what follows we propose a new augmented mixed formulation for our coupled problem. To do this, we suggest to enrich the variational formulation (3.9) with residual expressions arising from the equilibrium and constitutive equations, and the relation defining the vorticity in terms of the free fluid velocity. More precisely, we add the following terms:

$$\kappa_1(\mathrm{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\,,\,\mathrm{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \;=\; -\kappa_1(\mathbf{f}_{\mathrm{S}}\,,\,\mathrm{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}\,, \tag{4.1}$$

$$\kappa_2\left(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \;-\; \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}}\,,\,\mathbf{e}(\mathbf{v}_{\mathrm{S}})\right)_{\mathrm{S}} \;=\; 0\,, \tag{4.2}$$

$$\kappa_3\left(\boldsymbol{\gamma}_{\mathrm{S}} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathtt{t}})\,,\,\boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}} \;=\; 0\,, \tag{4.3}$$

for all $(\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}) \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})$, where κ_1, κ_2 and κ_3 are positive parameters to be specified later, and

$$\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \;:=\; \left\{\, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) \;:\; \mathbf{v}_{\mathrm{S}}|_{\Gamma_{\mathrm{S}}} = \mathbf{0} \,\right\}.$$

We notice here that (4.2) and (4.3) implicitly require now the velocity $\mathbf{u}_{\mathrm{S}}$ to live in the smaller space $\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$.

Let us now consider the global spaces:

$$\mathbf{X}_0 \;:=\; \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma)\,,$$

$$\mathbf{M}_0 \;:=\; L^2_0(\Omega_{\mathrm{D}}) \times \mathbb{R}\,,$$

endowed with the norms

$$\|\underline{\mathbf{s}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}} \;+\; |\mathbf{v}_{\mathrm{S}}|_{1,\Omega_{\mathrm{S}}} \;+\; \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,,\Omega_{\mathrm{D}}} \;+\; \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \;+ \|\xi\|_{1/2,\Sigma}\,,$$

and

$$\|\underline{\mathbf{q}}\|_{\mathbf{M}} \;:=\; \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \;+\; |\rho|,$$

for all $\underline{\mathbf{s}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbf{X}$, and $\underline{\mathbf{q}} := (q_{\mathrm{D}}, \rho) \in \mathbf{M}$, where $\mathbf{X}$ and $\mathbf{M}$ denote the product spaces defined respectively as $\mathbf{X}_0$ and $\mathbf{M}_0$, but considering the spaces $\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$, and $L^2(\Omega_{\mathrm{D}})$, instead of $\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$, and $L^2_0(\Omega_{\mathrm{D}})$. Then, defining the global unknowns as:

$$\underline{\mathbf{t}} \;:=\; (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda) \;\in\; \mathbf{X}_0, \qquad \underline{\mathbf{p}} \;:=\; (p_{\mathrm{D}}, \mu) \;\in\; \mathbf{M}_0,$$

the augmented mixed variational formulation reads: Find $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ such that

$$\begin{array}{rcll} \mathbf{A}(\underline{\mathbf{t}},\underline{\mathbf{s}}) + \mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{p}}) & = & \mathbf{F}(\underline{\mathbf{s}}) & \forall\, \underline{\mathbf{s}} \;:=\; (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbf{X}_0\,, \\ \mathbf{B}(\underline{\mathbf{t}},\underline{\mathbf{q}}) & = & \mathbf{G}(\underline{\mathbf{q}}) & \forall\, \underline{\mathbf{q}} \;:=\; (q_{\mathrm{D}}, \rho) \;\in\; \mathbf{M}_0\,, \end{array} \tag{4.4}$$

where

$$\mathbf{F}(\underline{\mathbf{s}}) \;:=\; (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \;-\kappa_1(\mathbf{f}_{\mathrm{S}}, \mathrm{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}\,, \qquad \mathbf{G}(\underline{\mathbf{q}}) \;:=\; -\,(f_{\mathrm{D}}, q_{\mathrm{D}})\,, \tag{4.5}$$

and $\mathbf{A}$ and $\mathbf{B}$ are the bounded bilinear forms defined by

$$\begin{aligned}\mathbf{A}(\underline{\mathbf{t}},\underline{\mathbf{s}}) \;:=\;& \mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}),(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})) \,+\, \mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}),(\boldsymbol{\varphi},\lambda))\\ &+\; \mathbf{b}((\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}),(\boldsymbol{\psi},\xi)) \,-\, \mathbf{c}((\boldsymbol{\varphi},\lambda),(\boldsymbol{\psi},\xi))\\ \mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{q}}) \;:=\;& -(\operatorname{div}\mathbf{v}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} \,+\, \rho\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_{\Sigma},\end{aligned}$$

with

$$\begin{aligned}\mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}),(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})) \;:=\;& \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} \,+\, \kappa_1\,(\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \,+\, (\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}}\\ &+\kappa_2\left(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}},\mathbf{e}(\mathbf{v}_{\mathrm{S}})\right)_{\mathrm{S}} + (\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\\ &+\kappa_3\left(\boldsymbol{\gamma}_{\mathrm{S}} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathtt{t}}),\boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}})_{\mathrm{S}} - (\boldsymbol{\sigma}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}},\end{aligned}$$

$$\mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}),(\boldsymbol{\psi},\xi)) \;:=\; \langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} \,-\, \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma},$$

and

$$\mathbf{c}((\boldsymbol{\varphi},\lambda),(\boldsymbol{\psi},\xi)) \;:=\; \langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} \,-\, \langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma} \,+\, \langle\boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma}.$$

We remark that the augmented mixed formulation (4.4) and the original fully-mixed scheme (3.9) are both represented by a twofold saddle-point operator equation. In other words, the bilinear forms $\mathbf{A}$ and $\mathcal{A}$ (see (3.11)) share the same doubly-mixed structure, which suggests to apply the same tools employed in Section 3, particularly Lemma 3.1, to analyze the well-posedness of (4.4). Nevertheless, the main advantages of this augmentation have to do with the fact that the resulting bilinear forms $\mathbf{B}$ and $\mathbf{b}$, not involving the rotation and not involving the stress tensor of the fluid, respectively, become less complicated than $\mathcal{B}$ and b, and hence the associated discrete inf-sup conditions are easier to satisfy. This feature is indeed confirmed below in Section 4.3 where we specify simpler finite element subspaces ensuring the well-posedness of the associated Galerkin scheme.

4.1 Analysis of the continuous augmented problem

In what follows, we apply the classical Babuška-Brezzi theory and Lemma 3.1 (cf. Section 3.2) to prove the well-posedness of (4.4). We start by establishing the inf-sup condition of the bilinear form $\mathbf{B}$:

Lemma 4.1 *There exists $\beta > 0$, such that*

$$\sup_{\underline{\mathbf{s}}\in\mathbf{X}_0\backslash\mathbf{0}} \frac{\mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{q}})}{\|\underline{\mathbf{s}}\|_{\mathbf{X}}} \;\geq\; \bar{\beta}\,\|\underline{\mathbf{q}}\|_{\mathbf{M}} \qquad \forall\,\underline{\mathbf{q}}\,\in\,\mathbf{M}_0\,. \tag{4.6}$$

Proof. It follows analogously to the proof of Lemma 3.2. We omit further details. □

Next, we apply again Lemma 3.1 to state the invertibility of $\mathbf{A}$ on the null space of the bilinear form $\mathbf{B}$, namely

$$\mathbf{V} \;:=\; \left\{\,\underline{\mathbf{s}}\,\in\,\mathbf{X}_0:\quad \mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{q}}) \,=\, 0 \quad \forall\,\underline{\mathbf{q}}\,\in\,\mathbf{M}_0\,\right\}.$$

To this end, we first recall that the well known Korn's inequality (see, for instance [38]) establishes the existence of a constant $\kappa_0 > 0$ such that

$$\|\mathbf{e}(\mathbf{v}_{\mathrm{S}})\|^2_{0,\Omega_{\mathrm{S}}} \,\ge\, \kappa_0\, |\mathbf{v}_{\mathrm{S}}|^2_{1,\Omega_{\mathrm{S}}} \qquad \forall\, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}). \tag{4.7}$$

On the other hand, it is easy to see from the definition of $\mathbf{B}$ that $\mathbf{V} \,=\, \mathbf{V}_1 \,\times\, \mathbf{V}_2$, where

$$\mathbf{V}_1 \,=\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \quad \text{and} \quad \mathbf{V}_2 \,=\, \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma)\,,$$

with

$$\tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \,:=\, \Big\{ \mathbf{v}_{\mathrm{D}} \,\in\, \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}): \quad \mathrm{div}\,\mathbf{v}_{\mathrm{D}} \,\in\, \mathbb{P}_0(\Omega_{\mathrm{D}}) \Big\}\,,$$

and

$$\tilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \,:=\, \Big\{ \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma): \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma \,=\, 0 \Big\}\,.$$

The invertibility of $\mathbf{A}$ on $\mathbf{V}$ is established next.

Lemma 4.2 *Assume that* $\kappa_1 \,>\, 0$, $0 \,<\, \kappa_2 \,<\, 4\,\nu$, *and* $0 \,<\, \kappa_3 \,<\, \kappa_0\,\kappa_2$, *with* κ_0 *the constant of the Korn's inequality* (4.7). *Then the bilinear form* $\mathbf{A}$ *is invertible.*

Proof. In what follows, we proceed similarly to the proof of Lemma 3.2, and verify that the bilinear forms $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, defining $\mathbf{A}$, satisfy the hypotheses of Lemma 3.1. First, we observe that, according to the diagonal character of $\mathbf{b}$, the inf-sup condition of $\mathbf{b}$ on $\mathbf{V}_1 \times \mathbf{V}_2$ holds if and only if

$$\sup_{\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\backslash \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \,\ge\, \bar{\beta}^{\mathrm{S}}_\Sigma\, \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \qquad \forall\, \boldsymbol{\psi} \,\in\, \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma)\,,$$

and

$$\sup_{\mathbf{v}_{\mathrm{D}} \in \tilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\backslash \mathbf{0}} \frac{\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \,\ge\, \bar{\beta}^{\mathrm{D}}_\Sigma\, \|\xi\|_{1/2,\Sigma} \qquad \forall\, \xi \,\in\, H^{1/2}(\Sigma)\,.$$

The first condition above has been already verified in Lemma 3.3, whereas the second one follows from the surjectivity of the operator $\boldsymbol{\tau} \,\to\, \boldsymbol{\tau}\mathbf{n}$, from $\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ to $\mathbf{H}^{-1/2}_{00}(\Sigma)$. We omit further details. Now, for the ellipticity of $\mathbf{a}$ on $\mathbf{V}_1$, we proceed similarly to [26, Section 3] (see also [27]). In fact, given $\boldsymbol{\zeta} = (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{V}_1$, we notice, after a simple computation, that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\ge\, & \left(\frac{1}{2\nu} - \frac{\kappa_2}{8\nu^2}\right) \|\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, \kappa_1\,\|\mathrm{div}\;\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, (\mathrm{K}^{-1}\mathbf{v}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \\
& +\, \frac{\kappa_3}{2}\|\boldsymbol{\eta}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, \frac{\kappa_2}{2}\,\|\mathbf{e}(\mathbf{v}_{\mathrm{S}})\|^2_{0,\Omega_{\mathrm{S}}} \,-\, \frac{\kappa_3}{2}\,|\mathbf{v}_{\mathrm{S}}|^2_{1,\Omega_{\mathrm{S}}}.
\end{aligned}$$

Hence, applying (2.4), (3.14), (3.15) and (4.7), and utilizing the assumptions on κ_1, κ_2 and κ_3, we find that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\ge\, & \frac{C_{\mathrm{S}}}{2}\min\left\{\left(\frac{1}{\nu} - \frac{\kappa_2}{4\nu^2}\right), \kappa_1\right\} \|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, \frac{\kappa_1}{2}\,\|\mathrm{div}\;\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \\
& +\, C_{\mathrm{K}}\,C_{\mathrm{D}}\,\|\mathbf{v}_{\mathrm{D}}\|^2_{\mathrm{div}\,,\Omega_{\mathrm{D}}} \,+\, \frac{\kappa_3}{2}\|\boldsymbol{\eta}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, \frac{1}{2}\,(\kappa_0\,\kappa_2 - \kappa_3)\,|\mathbf{v}_{\mathrm{S}}|^2_{1,\Omega_{\mathrm{S}}} \\
\ge\, & C\Big\{\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{\mathbf{div},\Omega_{\mathrm{S}}} \,+\, |\mathbf{v}_{\mathrm{S}}|^2_{1,\Omega_{\mathrm{S}}} \,+\, \|\boldsymbol{\eta}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \,+\, \|\mathbf{v}_{\mathrm{D}}\|^2_{\mathrm{div}\,,\Omega_{\mathrm{D}}}\Big\}\,.
\end{aligned}$$

We conclude the proof by observing that

$$\mathbf{c}((\boldsymbol{\psi},\xi),(\boldsymbol{\psi},\xi)) = \langle \boldsymbol{\psi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} \,\geq\, 0 \qquad \forall\,(\boldsymbol{\psi},\xi) \in \mathbf{V}_2 .$$

$\square$

According to the foregoing analysis, the well-posedness of (4.4) is stated as follows.

Theorem 4.3 *Assume that $\kappa_1 > 0$, $0 < \kappa_2 < 4\,\nu$, and $0 < \kappa_3 < \kappa_0\,\kappa_2$, with κ_0 the constant of the Korn's inequality* (4.7)*. Then, for each pair $(\mathbf{F},\mathbf{G}) \in \mathbf{X}_0' \times \mathbf{M}_0'$ there exists a unique $(\underline{\mathbf{t}},\underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ solution to* (4.4)*, and there exists a constant $C > 0$, independent of the solution, such that*

$$\|(\underline{\mathbf{t}},\underline{\mathbf{p}})\|_{\mathbf{X}\times\mathbf{M}} \;\leq\; C\left\{ \|\mathbf{F}\|_{\mathbf{X}_0'} \,+\, \|\mathbf{G}\|_{\mathbf{M}_0'} \right\}. \tag{4.8}$$

Proof. It follows from Lemmata 4.1 and 4.2, and a straightforward application of the classical Babuška-Brezzi theory. $\square$

4.2 The augmented mixed finite element method

In what follows, we define the Galerkin scheme of problem (4.4) and establish suitable hypotheses on the finite element subspaces ensuring later on its well-posedness. We first introduce arbitrary subspaces of $\mathbf{H}^1(\Omega_{\mathrm{S}})$ and $\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, namely

$$\mathbf{H}^1_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}^1(\Omega_{\mathrm{S}}) \quad \text{and} \quad \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \,:=\, \mathbf{H}^1_h(\Omega_{\mathrm{S}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}). \tag{4.9}$$

In addition, we consider again the subspaces from (3.23), and define the global discrete spaces as follows:

$$\begin{gathered}\mathbf{X}_{h,0} \,:=\, \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) \times \Lambda^{\mathrm{D}}_h(\Sigma)\,,\\ \mathbf{M}_{h,0} \,:=\, L_{h,0}(\Omega_{\mathrm{D}}) \times \mathbb{R}\,.\end{gathered} \tag{4.10}$$

In this way, the Galerkin scheme of (4.4) reads: Find $(\underline{\mathbf{t}}_h,\underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$ such that

$$\begin{array}{rcll}\mathbf{A}(\underline{\mathbf{t}}_h,\underline{\mathbf{s}}_h) \,+\, \mathbf{B}(\underline{\mathbf{s}}_h,\underline{\mathbf{p}}_h) & = & \mathbf{F}(\underline{\mathbf{s}}_h) & \forall\,\underline{\mathbf{s}}_h \in \mathbf{X}_{h,0}\,,\\ \mathbf{B}(\underline{\mathbf{t}}_h,\underline{\mathbf{q}}_h) & = & \mathbf{G}(\underline{\mathbf{q}}_h) & \forall\,\underline{\mathbf{q}}_h \,\in\, \mathbf{M}_{h,0}\,.\end{array} \tag{4.11}$$

Now, similarly as before, in order to guarantee the solvability of (4.11), we introduce suitable hypotheses on the finite element subspaces defining $\mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$. We begin by establishing the one ensuring the discrete inf-sup condition of $\mathbf{B}$ on $\mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$:

(H.4) *There exists $\widehat{\beta}_{\mathrm{D}} > 0$, independent of h, and there exists $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$, such that*

$$\sup_{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\backslash\mathbf{0}} \frac{(\operatorname{div}\mathbf{v}_{\mathrm{D},h},\, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\operatorname{div},\Omega_{\mathrm{D}}}} \;\geq\; \widehat{\beta}_{\mathrm{D}}\,\|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\, q_{\mathrm{D},h} \,\in\, L_{h,0}(\Omega_{\mathrm{D}})\,, \tag{4.12}$$

$$\boldsymbol{\psi}_0 \,\in\, \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) \quad \forall\, h \quad \textit{and} \quad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \neq 0\,. \tag{4.13}$$

Next, in order to have a more precise definition of the discrete kernel of $\mathbf{B}$, which is given by

$$\mathbf{V}_h := \left\{ \underline{\mathbf{s}} \in \mathbf{X}_{h,0} : \quad \mathbf{B}(\underline{\mathbf{s}}, \underline{\mathbf{q}}) = 0 \quad \forall \underline{\mathbf{q}} \in \mathbf{M}_{h,0} \right\},$$

we also assume:

(H.5) $\operatorname{div} \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$.

According to the above, it is not difficult to see that $\mathbf{V}_h$ can be decomposed as $\mathbf{V}_h = \mathbf{V}_{h,1} \times \mathbf{V}_{h,2}$, where

$$\mathbf{V}_{h,1} = \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{S}_h(\Omega_{\mathrm{S}}) \times \tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \quad \text{and} \quad \mathbf{V}_{h,2} = \tilde{\mathbf{\Lambda}}^{\mathrm{S}}_h(\Sigma) \times \Lambda^{\mathrm{D}}_h(\Sigma),$$

with

$$\tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) := \left\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) : \quad \operatorname{div} \mathbf{v}_{\mathrm{D}} \in \mathbb{P}_0(\Omega_{\mathrm{D}}) \right\},$$

and

$$\tilde{\mathbf{\Lambda}}^{\mathrm{S}}_h(\Sigma) := \left\{ \boldsymbol{\psi} \in \mathbf{\Lambda}^{\mathrm{S}}_h(\Sigma) : \quad \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \right\}.$$

The following hypothesis is necessary to prove the invertibility of $\mathbf{A}$ on $\mathbf{V}_h$.

(H.6) *There exist* $\widehat{\beta}^{\mathrm{S}}_\Sigma$, $\widehat{\beta}^{\mathrm{D}}_\Sigma > 0$, *independent of* h, *such that*

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}}) \setminus \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widehat{\beta}^{\mathrm{S}}_\Sigma \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \qquad \forall \boldsymbol{\psi}_h \in \tilde{\mathbf{\Lambda}}^{\mathrm{S}}_h(\Sigma) \tag{4.14}$$

$$\sup_{\mathbf{v}_{\mathrm{D},h} \in \tilde{\mathbf{H}}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \setminus \mathbf{0}} \frac{\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \geq \widehat{\beta}^{\mathrm{D}}_\Sigma \|\xi_h\|_{1/2,\Sigma} \qquad \forall \xi_h \in \Lambda^{\mathrm{D}}_h(\Sigma), \tag{4.15}$$

Observe, in particular, that (4.14) implies

$$\sup_{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \setminus \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widehat{\beta}^{\mathrm{S}}_\Sigma \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \qquad \forall \boldsymbol{\psi}_h \in \tilde{\mathbf{\Lambda}}^{\mathrm{S}}_h(\Sigma).$$

Notice, as previously announced, that hypotheses **(H.4)**, **(H.5)** and **(H.6)** are less demanding than hypotheses **(H.1)**, **(H.2)** and **(H.3)** in Section 3.3. This fact will be reflected in the next section, where we introduce the particular choices of finite elements subspaces.

The well-posedness of (4.11) and the associated Céa estimate are provided by the following theorem.

Theorem 4.4 *Assume that hypotheses* **(H.0)**, **(H.4)**, **(H.5)**, *and* **(H.6)** *hold. In addition, assume that* $\kappa_1 > 0$, $0 < \kappa_2 < 4\nu$, *and* $0 < \kappa_3 < \kappa_0\,\kappa_2$, *where* κ_0 *is the constant of Korn's inequality* (4.7). *Then, the Galerkin scheme* (4.11) *has a unique solution* $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$. *Moreover, there exist positive constants* C_1 *and* C_2, *independent of* h, *such that*

$$\|\underline{\mathbf{t}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{p}}_h\|_{\mathbf{M}} \leq C_1 \left\{ \|\mathbf{F}|_{\mathbf{X}_{h,0}}\|_{\mathbf{X}'_{h,0}} + \|\mathbf{G}|_{\mathbf{M}_{h,0}}\|_{\mathbf{M}'_{h,0}} \right\}, \tag{4.16}$$

and

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{M}} \leq C_2 \left\{ \inf_{\underline{\mathbf{s}}_h \in \mathbf{X}_{h,0}} \|\underline{\mathbf{t}} - \underline{\mathbf{s}}_h\|_{\mathbf{X}} + \inf_{\underline{\mathbf{q}}_h \in \mathbf{M}_{h,0}} \|\underline{\mathbf{p}} - \underline{\mathbf{q}}_h\|_{\mathbf{M}} \right\}. \tag{4.17}$$

Proof. The proof follows basically from Lemma 3.1 and the classical Babuška-Brezzi theory. In fact, it is straightforward to see, thanks to the diagonal character of $\mathbf{B}$, that the bilinear form $\mathbf{B}$ satisfies the discrete inf-sup condition on $\mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$ if **(H.4)** holds. Moreover, applying similar arguments to those utilized in the proof of Lemma 4.2, and having in mind the assumptions on κ_1, κ_2, and κ_3, and hypotheses **(H.5)** and **(H.6)**, we deduce that $\mathbf{a}$ is elliptic on $\mathbf{V}_{h,1}$ and $\mathbf{b}$ satisfies the discrete inf-sup condition on $\mathbf{V}_{h,1} \times \mathbf{V}_{h,2}$, which, together with Lemma 3.1, imply the invertibility of $\mathbf{A}$ on $\mathbf{V}_h$. $\square$

4.3 Particular choices of discrete spaces

Similarly to Section 3.4, we now introduce specific discrete spaces satisfying hypotheses **(H.0)**, **(H.4)**, **(H.5)** and **(H.6)** in 2D and 3D. In what follows, we make use of the same notations employed in Section 3.4 for the definition of the corresponding triangulations of Ω_{S} and Ω_{D}.

4.3.1 Raviart-Thomas elements in 2D

Let us assume the same hypotheses on the mesh given in Section 3.4.1, and consider the discrete spaces:

$$\begin{aligned}
\mathbf{H}_h^1(\Omega_{\mathrm{S}}) &:= \left\{ \mathbf{v}_h \in [\mathcal{C}(\bar{\Omega}_{\mathrm{S}})]^2 : \quad \mathbf{v}_h|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\
\mathbb{S}_h(\Omega_{\mathrm{S}}) &:= \left\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) : \boldsymbol{\eta}_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\
\mathbf{H}_h(\Omega_\star) &:= \left\{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \quad \tau_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\} \quad \star \in \{\mathrm{S},\mathrm{D}\}, \\
L_h(\Omega_\star) &:= \left\{ q_h \in L^2(\Omega_\star) : q_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\} \quad \star \in \{\mathrm{S},\mathrm{D}\}.
\end{aligned} \tag{4.18}$$

In addition, on the interface Σ we consider the subspaces introduced in Section 3.4.1, that is

$$\Lambda_h^{\mathrm{S}}(\Sigma) := \left\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \right\}, \tag{4.19}$$

and

$$\Lambda_h^{\mathrm{D}}(\Sigma) = \left\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h} \right\}. \tag{4.20}$$

Then, we define the global spaces $\mathbf{X}_{h,0}$ and $\mathbf{M}_{h,0}$ by combining (3.24), (4.9), (4.10), (4.18), (4.19), and (4.20).

We remark that these subspaces satisfy hypotheses **(H.0)**, **(H.4)**, **(H.5)** and **(H.6)**. Indeed, it is clear that hypotheses **(H.0)** and **(H.5)** hold true. In turn, observing that (4.12) and (4.13) coincide with (3.29) and (3.30), respectively, it follows, as explained in Section 3.4.1, that **(H.4)** holds true. Finally, the inf-sup conditions (4.14) and (4.15) are a direct consequence of the discrete inf-sup conditions (3.43) and (3.42), respectively, and [44, Theorem 5.1] (see Section 3.4.1 for details).

As a consequence of the above, we can establish the main results of this section. Their proofs are straightforward. In particular, the proof of Theorem 4.6 below certainly makes use of the approximation properties of the finite element subspaces employed (see, e.g. [4], [12], [15], [28], [38], [39]).

Theorem 4.5 *Let $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ be the unique solution of (4.4), and let $\mathbf{X}_{h,0}$ and $\mathbf{M}_{h,0}$ be the finite element subspaces defined by (3.24), (4.9) and (4.10), in terms of the specific discrete spaces given by (4.18), (4.19) and (4.20). In addition, assume that $\kappa_1 > 0$, $0 < \kappa_2 < 4\nu$, and $0 < \kappa_3 < \kappa_0 \kappa_2$, where κ_0 is the constant of Korn's inequality (4.7). Then, the Galerkin scheme (4.11) has a unique solution $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$, and there exist c_1, $c_2 > 0$, independent of h, such that*

$$\|(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \le c_1 \left\{ \|\mathbf{F}|_{\mathbf{X}_{h,0}}\|_{\mathbf{X}'_{h,0}} + \|\mathbf{G}|_{\mathbf{M}_{h,0}}\|_{\mathbf{M}'_{h,0}} \right\},$$

and

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{M}} \le c_2 \left\{ \inf_{\underline{\mathbf{s}}_h \in \mathbf{X}_{h,0}} \|\underline{\mathbf{t}} - \underline{\mathbf{s}}_h\|_{\mathbf{X}} + \inf_{\underline{\mathbf{q}}_h \in \mathbf{M}_{h,0}} \|\underline{\mathbf{p}} - \underline{\mathbf{q}}_h\|_{\mathbf{M}} \right\}. \tag{4.21}$$

Theorem 4.6 *Let $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ and $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$ be the unique solutions of the continuous and discrete problems (4.4) and (4.11), respectively. Assume that there exists $\delta \in (0,1]$ such that $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $\boldsymbol{\gamma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, and $\mathrm{div}\,\mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$. Then, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\lambda \in H^{1/2+\delta}(\Sigma)$, and there exists $C > 0$, independent of h, such that*

$$\begin{aligned} \|(\underline{\mathbf{t}}, \underline{\mathbf{p}}) - (\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \le C\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathrm{div}\,\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} \\ & + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\gamma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}. \end{aligned} \tag{4.22}$$

4.3.2 Raviart-Thomas elements in 3D

Let us now consider the discrete spaces:

$$\begin{aligned} \mathbf{H}^1_h(\Omega_{\mathrm{S}}) &:= \left\{ \mathbf{v}_h \in [\mathcal{C}(\bar{\Omega}_{\mathrm{S}})]^3 : \quad \mathbf{v}_h|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\ \mathbb{S}_h(\Omega_{\mathrm{S}}) &:= \left\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) : \boldsymbol{\eta}_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\ \mathbf{H}_h(\Omega_\star) &:= \left\{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \quad \tau_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\} \quad \star \in \{\mathrm{S}, \mathrm{D}\}, \\ L_h(\Omega_\star) &:= \left\{ q_h \in L^2(\Omega_\star) : q_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \right\} \quad \star \in \{\mathrm{S}, \mathrm{D}\}. \end{aligned} \tag{4.23}$$

In addition, on the interface Σ, we proceed as in Section 3.4.2 and introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}$, and define $\widehat{h}_\Sigma := \{\widehat{h}_K : K \in \Sigma_{\widehat{h}}\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, for the unknowns on the interface Σ we consider the discrete spaces (3.50) and (3.51), namely

$$\Lambda^{\mathrm{S}}_h(\Sigma) := \left\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_K \in P_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \psi_h = 0 \quad \text{on} \quad \partial\Sigma \right\}, \tag{4.24}$$

and

$$\Lambda^{\mathrm{D}}_h(\Sigma) = \left\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_K \in P_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \right\}. \tag{4.25}$$

In this way, the global spaces $\mathbf{X}_{h,0}$ and $\mathbf{M}_{h,0}$ are defined by combining (3.24), (4.9), (4.10), (4.23), (4.24), and (4.25).

Now, concerning the hypotheses **(H.0)**, **(H.4)**, **(H.5)** and **(H.6)**, we notice that **(H.0)**, **(H.4)** and **(H.5)** follow as explained in Section 4.3.1, whereas **(H.6)** is consequence of the inf-sup conditions (3.42) and (3.43), which follow from [31, Lemma 7.5] (see Section 3.4.2 for details). More precisely, [31, Lemma 7.5] establishes the existence of a constant $C_0 \in (0,1)$ such that for each pair $(h_\Sigma, h_{\widehat{\Sigma}})$ verifying $h_\Sigma \le C_0 h_{\widehat{\Sigma}}$, (3.42) and (3.43) are satisfied.

The main results of this section are collected in the following theorems. As before, we also remark here that Theorem 4.8 below makes use of the approximation properties of the finite element subspaces involved (see, e.g. [4], [12], [15], [28], [38], [39]).

Theorem 4.7 *Let* $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ *be the unique solution of* (4.4)*, and let* $\mathbf{X}_{h,0}$ *and* $\mathbf{M}_{h,0}$ *be the finite element subspaces defined by* (3.24)*,* (4.9) *and* (4.10)*, in terms of the specific discrete spaces given by* (4.23)*,* (4.24)*, and* (4.25)*. In addition, assume that* $\kappa_1 > 0$*,* $0 < \kappa_2 < 4\nu$*, and* $0 < \kappa_3 < \kappa_0 \kappa_2$*, where* κ_0 *is the constant of Korn's inequality* (4.7)*. Furthermore, suppose that* $h_\Sigma \le C_0 h_{\widehat{\Sigma}}$*. Then, the Galerkin scheme* (4.11) *has a unique solution* $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$*, and there exist* c_1*,* $c_2 > 0$*, independent of* h*, such that*

$$\|(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \le c_1 \left\{ \|\mathbf{F}|_{\mathbf{X}_{h,0}}\|_{\mathbf{X}'_{h,0}} + \|\mathbf{G}|_{\mathbf{M}_{h,0}}\|_{\mathbf{M}'_{h,0}} \right\},$$

and

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{M}} \le c_2 \left\{ \inf_{\underline{\mathbf{s}}_h \in \mathbf{X}_{h,0}} \|\underline{\mathbf{t}} - \underline{\mathbf{s}}_h\|_{\mathbf{X}} + \inf_{\underline{\mathbf{q}}_h \in \mathbf{M}_{h,0}} \|\underline{\mathbf{p}} - \underline{\mathbf{q}}_h\|_{\mathbf{M}} \right\}. \tag{4.26}$$

Theorem 4.8 *Let* $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X}_0 \times \mathbf{M}_0$ *and* $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_{h,0} \times \mathbf{M}_{h,0}$ *be the unique solutions of the continuous and discrete problems* (4.4) *and* (4.11)*, respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$*,* $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$*,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$*,* $\boldsymbol{\gamma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$*, and* $\mathrm{div}\,\mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$*. Then,* $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$*,* $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned} \|(\underline{\mathbf{t}}, \underline{\mathbf{p}}) - (\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \le C\,h^\delta \Big\{ & \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathrm{div}\,\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} \\ & + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\gamma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}. \end{aligned} \tag{4.27}$$

5 A posteriori error estimator

In this section we restrict ourselves to the two-dimensional case and derive a reliable and efficient residual-based a posteriori error estimate for our augmented mixed finite element scheme (4.11), with the discrete spaces introduced in Section 4.3.1. The extension to 3D should be quite straightforward. Most of the analysis employed in the proofs makes extensive use of the estimates derived in [6], [13] and [36] (see also [22]). We begin with some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$. let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{D}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_\star) := \{ e \in \mathcal{E}_h : e \subseteq \Gamma_\star \}$, $\mathcal{E}_h(\Omega_\star) := \{ e \in \mathcal{E}_h : e \subseteq \Omega_\star \}$, for each $\star \in \{\mathrm{S},\mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{ e \in \mathcal{E}_h : e \subseteq \Sigma \}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [\mathcal{C}(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_T)|_e - (q|_{T'})|_e$, where T and T' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1,n_2)^{\mathtt{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2,n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [\mathcal{C}(T)]^2$ and $\boldsymbol{\tau}|_T \in [\mathcal{C}(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v}\cdot\mathbf{t}_e]$ and $[\boldsymbol{\tau}\,\mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v}\cdot\mathbf{t}_e] := \{(\mathbf{v}|_T)|_e - (\mathbf{v}|_{T'})|_e\}\cdot\mathbf{t}_e$ and $[\boldsymbol{\tau}\,\mathbf{t}_e] := \{(\boldsymbol{\tau}|_T)|_e - (\boldsymbol{\tau}|_{T'})|_e\}\,\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for sufficiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1,v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}\,q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathtt{t}},$$

$$\mathrm{rot}\,\mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathtt{t}}.$$

Next, for the sake of simplicity, in this section we replace the augmented formulation (4.4) by the equivalent one arising from the utilization of the decomposition (3.4). In other words, we drop the explicit unknown $\mu \in \mathbb{R}$ and keep it implicitly by redefining the stress $\boldsymbol{\sigma}$ as an unknown in $\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}})$. In this way, the augmented mixed formulation reduces to: Find $(\underline{\mathbf{t}},\underline{\mathbf{p}}) := ((\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\varphi},\lambda)\,,\,p_{\mathrm{D}})$ $\in \mathbf{X}\times\mathbf{M}_0$, such that

$$\begin{aligned} \mathbf{A}(\underline{\mathbf{t}},\underline{\mathbf{s}}) + \mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{p}}) &= \mathbf{F}(\underline{\mathbf{s}}) \qquad \forall\,\underline{\mathbf{s}} := (\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi},\xi) \in \mathbf{X}\,, \\ \mathbf{B}(\underline{\mathbf{t}},\underline{\mathbf{q}}) &= \mathbf{G}(\underline{\mathbf{q}}) \qquad \forall\,\underline{\mathbf{q}} := q_{\mathrm{D}} \in \mathbf{M}_0\,. \end{aligned} \tag{5.1}$$

where

$$\mathbf{X} := \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\times\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\times\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})\times\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\times\mathbf{H}^{1/2}_{00}(\Sigma)\times H^{1/2}(\Sigma)\,,$$

and

$$\mathbf{M}_0 := L^2_0(\Omega_{\mathrm{D}})\,.$$

Here, $\mathbf{B}$ is redefined by suppressing the last term, that is $\mathbf{B}(\underline{\mathbf{s}},\underline{\mathbf{q}}) := -(\mathrm{div}\,\mathbf{v}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}$, for all $(\underline{\mathbf{s}},\underline{\mathbf{q}}) \in \mathbf{X}\times\mathbf{M}_0$. Consequently, the equivalent discrete problem is defined as follows: Find $(\underline{\mathbf{t}}_h,\underline{\mathbf{p}}_h)$ $:= ((\boldsymbol{\sigma}_{\mathrm{S},h},\mathbf{u}_{\mathrm{S},h},\boldsymbol{\gamma}_{\mathrm{S},h},\mathbf{u}_{\mathrm{D},h},\boldsymbol{\varphi}_h,\lambda_h)\,,\,p_{\mathrm{D},h}) \in \mathbf{X}_h\times\mathbf{M}_{h,0}$ such that

$$\begin{aligned} \mathbf{A}(\underline{\mathbf{t}}_h,\underline{\mathbf{s}}_h) + \mathbf{B}(\underline{\mathbf{s}}_h,\underline{\mathbf{p}}_h) &= \mathbf{F}(\underline{\mathbf{s}}_h) \qquad \forall\,\underline{\mathbf{s}}_h := (\boldsymbol{\tau}_{\mathrm{S},h},\mathbf{v}_{\mathrm{S},h},\boldsymbol{\eta}_{\mathrm{S},h},\mathbf{v}_{\mathrm{D},h},\boldsymbol{\psi}_h,\xi_h) \in \mathbf{X}_h\,, \\ \mathbf{B}(\underline{\mathbf{t}}_h,\underline{\mathbf{q}}_h) &= \mathbf{G}(\underline{\mathbf{q}}_h) \qquad \forall\,\underline{\mathbf{q}}_h := q_{\mathrm{D},h} \in \mathbf{M}_{h,0}\,, \end{aligned} \tag{5.2}$$

where

$$\mathbf{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}})\times\mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\times\mathbb{S}_h(\Omega_{\mathrm{S}})\times\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\times\boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)\times\Lambda^{\mathrm{D}}_h(\Sigma)\,,$$

and

$$\mathbf{M}_{h,0} := L_{h,0}(\Omega_{\mathrm{D}})\,.$$

We recall that $\mathbf{X}_h$ and $\mathbf{M}_{h,0}$ are defined in terms of the discrete spaces introduced in Section 4.3.1. In addition, thanks to the equivalence between (4.4) and (5.1) (equivalently (4.11) and (5.2)), it is clear that both problems are well-posed and satisfy the corresponding continuous dependence inequalities.

On the other hand, let $(\underline{\mathbf{t}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M}_0$ and $(\underline{\mathbf{t}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_{h,0}$, be the unique solutions of (5.1) and (5.2), respectively. Then we introduce the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{5.3}$$

where for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned}
\Theta_{\mathrm{S},T}^2 &:= \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + \left\|\boldsymbol{\gamma}_{\mathrm{S},h} - \frac{1}{2}(\nabla \mathbf{u}_{\mathrm{S},h} - (\nabla \mathbf{u}_{\mathrm{S},h})^{\mathtt{t}})\right\|_{0,T}^2 + \left\|\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) - \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\right\|_{0,T}^2 \\
&+ \|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}\|_{0,T}^2 + h_T^2 \left\|\mathbf{rot}\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\right)\right\|_{0,T}^2 + h_T^2 \left\|\nabla \mathbf{u}_{\mathrm{S},h} - \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \boldsymbol{\gamma}_{\mathrm{S},h}\right\|_{0,T}^2 \\
&+ \sum_{e\,\in\,\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e \left\|\left[\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\right)\mathbf{t}\right]\right\|_{0,e}^2 + \sum_{e\,\in\,\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \left\|\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\right)\mathbf{t}\right\|_{0,e}^2 \\
&+ \sum_{e\,\in\,\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\|\left(\boldsymbol{\gamma}_{\mathrm{S},h}\mathbf{t} + \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\mathbf{t}\right) + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \\
&+ \sum_{e\,\in\,\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\{ \|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \pi_1^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|_{0,e}^2 + \|\boldsymbol{\varphi}_h + \mathbf{u}_{\mathrm{S},h}\|_{0,e}^2 \right\},
\end{aligned} \tag{5.4}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}
\Theta_{\mathrm{D},T}^2 &:= \|f_{\mathrm{D}} - \operatorname{div}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\|\operatorname{rot}(\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_T^2\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\|_{0,e}^2 + h_e\,\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 + h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \right\} \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e\,\|[\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e\,\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}\|_{0,e}^2 .
\end{aligned}$$

Hereafter, π_1 denotes the only frictional constant in (2.5), and the expressions $\boldsymbol{\varphi}_h'$ and λ_h' stand for the tangential derivatives of $\boldsymbol{\varphi}_h$ and λ_h, respectively, along Σ.

The main result of this section is stated as follows.

Theorem 5.1 *There exist positive constants $C_{\mathtt{rel}}$ and $C_{\mathtt{eff}}$, independent of h, such that*

$$C_{\mathtt{eff}}\,\Theta \leq \|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{M}} \leq C_{\mathtt{rel}}\,\Theta\,. \tag{5.5}$$

The efficiency of Θ (lower bound in (5.5)) is proved below in Section 5.2, whereas the reliability (upper bound in (5.5)) is proved next in Section 5.1.

5.1 Reliability of the a posteriori error estimator

We begin the derivation of the upper bound in (5.5) by recalling that the continuous dependence result given by (4.8) (after the corresponding modifications on the continuous spaces), is equivalent to the global inf–sup condition for the continuous formulation (5.1). Then, applying this estimate to the error $(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h) \in \mathbf{X} \times \mathbf{M}_0$, we obtain

$$\|(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \leq C \sup_{(\underline{\mathbf{s}},\underline{\mathbf{q}})\in\mathbf{X}\times\mathbf{M}} \frac{|R(\underline{\mathbf{s}},\underline{\mathbf{q}})|}{\|(\underline{\mathbf{s}},\underline{\mathbf{q}})\|_{\mathbf{X}\times\mathbf{M}}}\,, \tag{5.6}$$

where $R : \mathbf{X} \times \mathbf{M}_0 \to \mathbb{R}$ is the residual functional given by

$$R(\underline{\mathbf{s}},\underline{\mathbf{q}}) := \mathbf{A}(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\mathbf{s}}) + \mathbf{B}(\underline{\mathbf{s}}, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h) + \mathbf{B}(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\mathbf{q}}) \quad \forall\, (\underline{\mathbf{s}},\underline{\mathbf{q}}) \in \mathbf{X} \times \mathbf{M}_0.$$

More precisely, according to (5.1) and the definitions of $\mathbf{A}$ and $\mathbf{B}$, we find after a simple computation that for any $(\underline{\mathbf{s}},\underline{\mathbf{q}}) := ((\boldsymbol{\tau}_{\rm S}, \mathbf{v}_{\rm S}, \boldsymbol{\eta}_{\rm S}, \mathbf{v}_{\rm D}, \boldsymbol{\psi}, \xi), q_{\rm D}) \in \mathbf{X} \times \mathbf{M}_0$, there holds

$$R(\underline{\mathbf{s}},\underline{\mathbf{q}}) := R_1(\boldsymbol{\tau}_{\rm S}) + R_2(\mathbf{v}_{\rm D}) + R_3(\mathbf{v}_{\rm S}) + R_4(\boldsymbol{\eta}_{\rm S}) + R_5(q_{\rm D}) + R_6(\boldsymbol{\psi}) + R_7(\xi)\,,$$

where

$$\begin{aligned}
R_1(\boldsymbol{\tau}_{\rm S}) &:= -\kappa_1\,(\mathbf{f}_{\rm S} + \mathbf{div}\boldsymbol{\sigma}_{{\rm S},h}, \mathbf{div}\boldsymbol{\tau}_{\rm S})_{\rm S} - \frac{1}{2\nu}\,(\boldsymbol{\sigma}^{\tt d}_{{\rm S},h}, \boldsymbol{\tau}^{\tt d}_{\rm S})_{\rm S} \\
&\qquad - (\mathbf{div}\boldsymbol{\tau}_{\rm S}, \mathbf{u}_{{\rm S},h})_{\rm S} - (\boldsymbol{\gamma}_{{\rm S},h}, \boldsymbol{\tau}_{\rm S})_{\rm S} - \langle \boldsymbol{\tau}_{\rm S}\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma\,, \\
R_2(\mathbf{v}_{\rm D}) &:= -({\rm K}^{-1}\mathbf{u}_{{\rm D},h}, \mathbf{v}_{\rm D})_{\rm D} + \langle \mathbf{v}_{\rm D}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma + ({\rm div}\,\mathbf{v}_{\rm D}, p_{{\rm D},h})_{\rm D}\,, \\
R_3(\mathbf{v}_{\rm S}) &:= (\mathbf{f}_{\rm S} + \mathbf{div}\boldsymbol{\sigma}_{{\rm S},h}, \mathbf{v}_{\rm S})_{\rm S} - \kappa_2\left(\mathbf{e}(\mathbf{u}_{{\rm S},h}) - \frac{1}{2\nu}\boldsymbol{\sigma}^{\tt d}_{{\rm S},h}, \mathbf{e}(\mathbf{v}_{\rm S})\right)_{\rm S}, \\
R_4(\boldsymbol{\eta}_{\rm S}) &:= \frac{1}{2}\left(\boldsymbol{\sigma}_{{\rm S},h} - \boldsymbol{\sigma}^{\tt t}_{{\rm S},h}, \boldsymbol{\eta}_{\rm S}\right)_{\rm S} - \kappa_3\left(\boldsymbol{\gamma}_{{\rm S},h} - \frac{1}{2}(\nabla\mathbf{u}_{{\rm S},h} - (\nabla\mathbf{u}_{{\rm S},h})^{\tt t}), \boldsymbol{\eta}_{\rm S}\right)_{\rm S}, \\
R_5(q_{\rm D}) &:= -(f_{\rm D} - {\rm div}\,\mathbf{u}_{{\rm D},h}, q_{\rm D})_{\rm D}\,, \\
R_6(\boldsymbol{\psi}) &:= -\langle \boldsymbol{\sigma}_{{\rm S},h}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \pi_1^{-1}\langle \boldsymbol{\varphi}_h\cdot\mathbf{t}, \boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,, \\
R_7(\xi) &:= \langle \mathbf{u}_{{\rm D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma.
\end{aligned}$$

Hence, the supremum in (5.6) can be bounded in terms of R_i, $i = 1, \ldots, 7$, which yields

$$\begin{aligned}
\|(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}} \leq C \Big\{ &\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\rm S})'} + \|R_2\|_{\mathbf{H}_{\Gamma_{\rm D}}({\rm div};\Omega_{\rm D})'} + \|R_3\|_{\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})'} \\
&+ \|R_4\|_{\mathbb{L}^2_{\rm skew}(\Omega_{\rm S})'} + \|R_5\|_{L^2_0(\Omega_{\rm D})'} + \|R_6\|_{\mathbf{H}^{1/2}_{00}(\Sigma)'} + \|R_7\|_{H^{1/2}(\Sigma)'} \Big\}.
\end{aligned} \tag{5.7}$$

Throughout the rest of this section, we provide suitable upper bounds for each one of the terms on the right hand side of (5.7). We start with the following lemma, which is a direct consequence of the Cauchy-Schwarz inequality.

Lemma 5.2 *There exist C_3, $C_4 > 0$, independent of h, such that*

$$\|R_3\|_{\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})'} \leq C_3 \left\{ \sum_{T\in\mathcal{T}_h^{\rm S}} \|\mathbf{f}_{\rm S} + \mathbf{div}\boldsymbol{\sigma}_{{\rm S},h}\|^2_{0,T} + \left\|\mathbf{e}(\mathbf{u}_{{\rm S},h}) - \frac{1}{2\nu}\boldsymbol{\sigma}^{\tt d}_{{\rm S},h}\right\|^2_{0,T} \right\}^{1/2},$$

and

$$\|R_4\|_{\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})'} \leq \; C_4 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \left\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}\right\|_{0,T}^2 \; + \; \left\|\boldsymbol{\gamma}_{\mathrm{S},h} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S},h} - (\nabla\mathbf{u}_{\mathrm{S},h})^{\mathtt{t}})\right\|_{0,T}^2 \right\}^{1/2} .$$

In addition, there holds

$$\|R_5\|_{L^2_0(\Omega_{\mathrm{D}})'} \; \leq \; \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \|f_{\mathrm{D}} - \operatorname{div}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \right\}^{1/2} .$$

Next, we establish the estimates for the terms acting on the interface Σ. Their proofs can be found in [36, Lemma 3.2].

Lemma 5.3 *There exist C_6, $C_7 \, > \, 0$, independent of h, such that*

$$\|R_6\|_{\mathbf{H}^{1/2}_{00}(\Sigma)'} \leq C_6 \left\{ \sum_{e\,\in\,\mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \pi_1^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|_{0,e}^2 \right\}^{1/2}$$

and

$$\|R_7\|_{H^{1/2}(\Sigma)'} \leq C_7 \left\{ \sum_{e\,\in\,\mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \right\}^{1/2} .$$

The upper bound for $\|R_2\|_{\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})'}$ is provided next. Its proof can be found in [36, Lemma 3.9].

Lemma 5.4 *There exists $C_2 \, > \, 0$, independent of h, such that*

$$\|R_2\|_{\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})'} \; \leq \; C_2 \; \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \widehat{\Theta}^2_{\mathrm{D},T} \right\}^{1/2} , \tag{5.8}$$

where, for each $T \, \in \, \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned} \widehat{\Theta}^2_{\mathrm{D},T} \; &:= \; h_T^2 \,\|\mathrm{rot}\,(\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \; + \; h_T^2 \,\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\ &+ \quad \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e \, \left\|[\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\right\|_{0,e}^2 \; + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e \, \left\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}\right\|_{0,e}^2 \\ &+ \quad \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\{ \, \left\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\right\|_{0,e}^2 \; + \; \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}. \end{aligned}$$

Our next goal is to derive the upper bound for $\|R_1\|_{\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}})'}$. To do this, we first need to recall some known results. We begin with the following lemma establishing the existence of a stable Helmholtz decomposition for $\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, whose proof can be found, for instance, in [36, Lemma 3.3].

Lemma 5.5 *For each $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ there exist $\boldsymbol{\rho} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ such that there hold $\boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\rho} + \mathbf{curl}\,\boldsymbol{\chi}$ in Ω_{S}, and*

$$\|\boldsymbol{\rho}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \le C\,\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}},$$

where C is a positive constant independent of $\boldsymbol{\tau}_{\mathrm{S}}$.

We now recall the definition of $\mathbf{H}_h(\Omega_{\mathrm{S}})$ in Section (4.3.1), and let $\Pi_h : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{H}_h(\Omega_{\mathrm{S}})$ be the Raviart-Thomas interpolation operator (see [12, 28]), which, given $\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$, is characterized by the identity

$$\int_e \Pi_h\mathbf{v}\cdot\mathbf{n} = \int_e \mathbf{v}\cdot\mathbf{n} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^{\mathrm{S}}. \tag{5.9}$$

Note that, as a consequence of (5.9), there holds

$$\operatorname{div}(\Pi_h\mathbf{v}) = \mathcal{P}_h(\operatorname{div}\mathbf{v}), \tag{5.10}$$

where $\mathcal{P}_h$ is the $L^2(\Omega_{\mathrm{S}})$-orthogonal projector onto the piecewise constant functions on Ω_{S}. A tensor version of Π_h, say $\boldsymbol{\Pi}_h : \mathbb{H}^1(\Omega_{\mathrm{S}}) \to \mathbb{H}_h(\Omega_{\mathrm{S}})$, which is defined row-wise by Π_h, and a vector version of $\mathcal{P}_h$, say $\mathbf{P}_h$, which is the $\mathbf{L}^2(\Omega_{\mathrm{S}})$-orthogonal projector onto the piecewise constant vectors on Ω_{S}, might also be required. The local approximation properties of Π_h (and hence of $\boldsymbol{\Pi}_h$) are established in the following lemma. Its proof can be found in [12].

Lemma 5.6 *There exist constants c_1, $c_2 > 0$, independent of h, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ there hold*

$$\|\mathbf{v} - \Pi_h\mathbf{v}\|_{0,T} \le c_1\,h_T\,\|\mathbf{v}\|_{1,T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}},$$

and

$$\|\mathbf{v}\cdot\mathbf{n} - \Pi_h\mathbf{v}\cdot\mathbf{n}\|_{0,e} \le c_2\,h_e^{1/2}\,\|\mathbf{v}\|_{1,T_e} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^{\mathrm{S}},$$

where T_e is a triangle of $\mathcal{T}_h^{\mathrm{S}}$ containing e on its boundary.

In turn, we let $I_h : H^1(\Omega_{\mathrm{S}}) \to H_h^1(\Omega_{\mathrm{S}})$ be the classical Clément interpolation operator (see [13]), where

$$H_h^1(\Omega_{\mathrm{S}}) := \{v \in \mathcal{C}(\bar{\Omega}_{\mathrm{S}}) : \quad v|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\}.$$

In what follows, we will also make use of a vector version of I_h, say $\mathbf{I}_h : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{H}_h^1(\Omega_{\mathrm{S}}) := [H_h^1(\Omega_{\mathrm{S}})]^2$, which is defined component-wise by I_h. The following lemma establishes the local approximation properties of I_h (and hence of $\mathbf{I}_h$). Its proof can be found in [17].

Lemma 5.7 *There exist positive constants c_3, $c_4 > 0$, independent of h, such that*

$$\|v - I_h v\|_{0,T} \le c_3\,h_T\,\|v\|_{1,\Delta_{\mathrm{S}}(T)} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}},$$

and

$$\|v - I_h v\|_{0,e} \le c_4\,h_e^{1/2}\,\|v\|_{1,\Delta_{\mathrm{S}}(e)} \qquad \forall\, e \in \mathcal{E}_h,$$

for all $v \in H^1(\Omega_{\mathrm{S}})$, where

$$\Delta_{\mathrm{S}}(T) := \cup\{T' \in \mathcal{T}_h^{\mathrm{S}} : \quad T' \cap T \neq \phi\} \quad \text{and} \quad \Delta_{\mathrm{S}}(e) := \cup\{T' \in \mathcal{T}_h^{\mathrm{S}} : \quad T' \cap e \neq \phi\}.$$

Now, regarding the upper bound for R_1, let us consider the following decomposition:

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) \,=\, R_1^1(\boldsymbol{\tau}_{\mathrm{S}}) \,+\, R_1^2(\boldsymbol{\tau}_{\mathrm{S}})$$

where

$$R_1^1(\boldsymbol{\tau}_{\mathrm{S}}) \,:=\, -\kappa_1\,(\mathbf{f}_{\mathrm{S}} + \mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}),$$

and

$$R_1^2(\boldsymbol{\tau}_{\mathrm{S}}) \,:=\, -\left(\frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h} \,+\, \boldsymbol{\gamma}_{\mathrm{S},h}\,, \boldsymbol{\tau}_{\mathrm{S}}\right)_{\mathrm{S}} - (\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S},h})_{\mathrm{S}} - \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_{\Sigma}.$$

Then, we have the following preliminary estimates.

Lemma 5.8 *Let $\boldsymbol{\rho} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$. Then there exist $C, \tilde{C} > 0$, independent of h, such that*

$$|R_1^1(\boldsymbol{\rho} - \boldsymbol{\Pi}_h(\boldsymbol{\rho}))| \,\le\, C \sum_{T\,\in\,\mathcal{T}_h^{\mathrm{S}}} \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}\|\boldsymbol{\rho}\|_{1,T}\,, \tag{5.11}$$

and

$$\begin{aligned}|R_1^2((\boldsymbol{\rho} - \boldsymbol{\Pi}_h(\boldsymbol{\rho}))| \,\le\, \tilde{C}\Bigg\{ &\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} h_T \left\|\nabla\mathbf{u}_{\mathrm{S},h} - \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h} \,-\, \boldsymbol{\gamma}_{\mathrm{S},h}\right\|_{0,T} \|\boldsymbol{\rho}\|_{1,T} \\ &+ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\,\|\boldsymbol{\varphi}_h + \mathbf{u}_{\mathrm{S},h}\|_{0,e}\,\|\boldsymbol{\rho}\|_{1,T_e}\Bigg\},\end{aligned} \tag{5.12}$$

where T_e is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. Let $\boldsymbol{\rho} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$. We first observe, using the vector version of (5.10), that for each $T \in \mathcal{T}_h^{\mathrm{S}}$ there holds

$$\|\mathbf{div}(\boldsymbol{\rho} - \boldsymbol{\Pi}_h(\boldsymbol{\rho}))\|_{0,T} = \|\mathbf{div}\,\boldsymbol{\rho} - \mathbf{P}_h(\mathbf{div}\,\boldsymbol{\rho})\|_{0,T} \,\le\, C\,\|\mathbf{div}\,\boldsymbol{\rho}\|_{0,T} \le C\|\boldsymbol{\rho}\|_{1,T}.$$

Hence, (5.11) follows from a straightforward application of the Cauchy–Schwarz inequality and the continuity of $\mathbf{P}_h$, whereas (5.12) is derived through a slight modification of the proof of [36, Lemma 3.6]. □

Lemma 5.9 *Let $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$. Then there exists $C > 0$, independent of h, such that*

$$\begin{aligned}\left|R_1^2(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h(\boldsymbol{\chi})))\right| \,\le\, C\Bigg\{ &\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} h_T \left\|\mathbf{rot}\,\left(\boldsymbol{\gamma}_{\mathrm{S},h} \,+\, \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\right\|_{0,T} \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(T)} \\ + &\sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e^{1/2} \left\|\left[\left(\boldsymbol{\gamma}_{\mathrm{S},h} \,+\, \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\,\mathbf{t}\right]\right\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)} \\ + &\sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e^{1/2} \left\|\left(\boldsymbol{\gamma}_{\mathrm{S},h} \,+\, \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\,\mathbf{t}\right\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)} \\ + &\sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2} \left\|\left(\boldsymbol{\gamma}_{\mathrm{S},h} \,+\, \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\,\mathbf{t} \,+\, \boldsymbol{\varphi}_h'\right\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)}\Bigg\}.\end{aligned}$$

Proof. This result follows basically from [36, Lemma 3.6]. □

The upper bound for R_1 can now be established.

Lemma 5.10 *There exists $C_1 > 0$, independent of h, such that*

$$\|R_1\|_{\mathbb{H}(\mathbf{div},\Omega_S)'} \,\leq\, C_1 \left\{ \sum_{T\in\mathcal{T}_h^S} \widetilde{\Theta}_{S,T}^2 \right\}, \tag{5.13}$$

where, for each $T \in \mathcal{T}_h^S$:

$$\begin{aligned}
\widetilde{\Theta}_{S,T}^2 &:= \|\mathbf{f}_S + \mathbf{div}\boldsymbol{\sigma}_{S,h}\|_{0,T}^2 + h_T^2 \left\|\nabla\mathbf{u}_{S,h} - \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}} - \boldsymbol{\gamma}_{S,h}\right\|_{0,T}^2 + h_T^2 \left\|\mathbf{rot}\left(\boldsymbol{\gamma}_{S,h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}}\right)\right\|_{0,T}^2 \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\varphi}_h + \mathbf{u}_{S,h}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\|\left(\boldsymbol{\gamma}_{S,h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}}\right)\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e \left\|\left[\left(\boldsymbol{\gamma}_{S,h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}}\right)\mathbf{t}\right]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e \left\|\left(\boldsymbol{\gamma}_{S,h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}}\right)\mathbf{t}\right\|_{0,e}^2 .
\end{aligned}$$

Proof. Given $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div};\Omega_S)$, we let $\boldsymbol{\rho} \in \mathbb{H}^1(\Omega_S)$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_S)$ be the elements provided by Lemma 5.5, satisfying $\boldsymbol{\tau}_S = \boldsymbol{\rho} + \mathbf{curl}\,\boldsymbol{\chi}$ in Ω_S, and

$$\|\boldsymbol{\rho}\|_{1,\Omega_S} + \|\boldsymbol{\chi}\|_{1,\Omega_S} \,\leq\, C\,\|\boldsymbol{\tau}_S\|_{\mathbf{div};\Omega_S}\,. \tag{5.14}$$

Then, we set $\boldsymbol{\tau}_{S,h} := \boldsymbol{\Pi}_h(\boldsymbol{\rho}) + \mathbf{curl}\,(\mathbf{I}_h(\boldsymbol{\chi}))$, which can be seen as a discrete Helmholtz decomposition of $\boldsymbol{\tau}_{S,h}$. Then, thanks to the Galerkin orthogonality result we know that $R_1(\boldsymbol{\tau}_{S,h}) = 0$, which yields

$$R_1(\boldsymbol{\tau}_S) \,=\, R_1(\boldsymbol{\tau}_S - \boldsymbol{\tau}_{S,h}) \,=\, R_1(\boldsymbol{\rho} - \boldsymbol{\Pi}_h(\boldsymbol{\rho})) \,+\, R_1\big(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h(\boldsymbol{\chi}))\big)\,.$$

In this way, applying Lemmata 5.8 and 5.9, observing that the number of triangles in $\Delta_S(T)$ and $\Delta_S(e)$ are bounded, and using the estimate (5.14), we obtain the result. $\square$

We end this section by observing that the reliability estimate (cf. Theorem 5.1) is a direct consequence of Lemmas 5.2, 5.3, 5.4 and 5.10.

5.2 Efficiency of the a posteriori estimator

We now aim to prove the efficiency of Θ, that is, the lower bound in (5.5). We begin with the estimates for the zero order terms appearing in the definition of $\Theta_{S,T}^2$ and $\Theta_{D,T}^2$.

Lemma 5.11 *There hold*

$$\begin{aligned}
\|\mathbf{f}_S + \mathbf{div}\,\boldsymbol{\sigma}_{S,h}\|_{0,T} &\leq \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div},T} \quad \forall\, T\in\mathcal{T}_h^S\,,\\
\|f_D - \mathrm{div}\;\mathbf{u}_{D,h}\|_{0,T} &\leq \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathrm{div}\,,T} \quad \forall\, T\in\mathcal{T}_h^D\,,\\
\|\boldsymbol{\sigma}_{S,h} - \boldsymbol{\sigma}_{S,h}^{\mathtt{t}}\|_{0,T} &\leq C_1\,\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div},T} \quad \forall\, T\in\mathcal{T}_h^S\,,\\
\left\|\mathbf{e}(\mathbf{u}_{S,h}) - \frac{1}{2\nu}\boldsymbol{\sigma}_{S,h}^{\mathtt{d}}\right\|_{0,T}^2 &\leq C_2\left\{|\mathbf{u}_S - \mathbf{u}_{S,h}|_{1,T}^2 + \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,T}^2\right\} \quad \forall\, T\in\mathcal{T}_h^S\,,
\end{aligned}$$

and

$$\left\|\boldsymbol{\gamma}_{S,h} - \frac{1}{2}\big(\nabla\mathbf{u}_{S,h} - (\nabla\mathbf{u}_{S,h})^{\mathtt{t}}\big)\right\|_{0,T}^2 \leq C_3\left\{\|\boldsymbol{\gamma}_S - \boldsymbol{\gamma}_{S,h}\|_{0,T}^2 + |\mathbf{u}_S - \mathbf{u}_{S,h}|_{1,T}^2\right\} \quad \forall\, T\in\mathcal{T}_h^S\,,$$

where C_1, C_2, $C_3 > 0$ are independent of h.

Proof. This result follows by using the relations $\mathbf{f}_{\mathrm{S}} = -\,\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}$, $f_{\mathrm{D}} = \operatorname{div}\mathbf{u}_{\mathrm{D}}$, $\mathbf{e}(\mathbf{u}_{\mathrm{S}}) = \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}}$, $\boldsymbol{\gamma}_{\mathrm{S}} = \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathtt{t}})$, and the symmetry of $\boldsymbol{\sigma}_{\mathrm{S}}$. We omit further details. $\square$

In turn, the following lemma is a straightforward application of the triangle inequality and the relation $\frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}} = \nabla\mathbf{u}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S}}$ in Ω_{S}.

Lemma 5.12 *There exists $C > 0$, independent of h, such that for each $T \in \mathcal{T}_h^{\mathrm{S}}$ there holds*

$$h_T^2 \left\| \nabla\mathbf{u}_{\mathrm{S},h} - \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \boldsymbol{\gamma}_{\mathrm{S},h} \right\|_{0,T}^2 \le C\, h_T^2 \left\{ |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 + \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2 \right\}.$$

The derivation of the upper bounds of the remaining terms defining the global a posteriori error estimator proceeds similarly to [36] (see also [5]), using known results mainly from [13], [14], and [25], and applying inverse inequalities and the localization technique based on element-bubble and edge-bubble functions. In particular, the following lemma summarizes the upper bounds of nine terms defining $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$.

Lemma 5.13 *There exist positive constants C_i, $i \in \{1, ..., 9\}$, independent of h, such that*

a) $h_T^2\, \|\operatorname{rot}(\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \le C_1\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}$,

b) $h_T^2 \left\| \mathbf{rot}\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}\right)\right\|_{0,T}^2$
$\le C_2 \left\{ \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \right\} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}$,

c) $h_T^2\, \|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \le C_3 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \right\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}$,

d) $h_e\, \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \le C_4 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_e\, \|\lambda - \lambda_h\|_{0,e}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Sigma)$,
where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,

e) $h_e\, \|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2$
$\le C_5 \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\operatorname{div}(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_e\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}$, *for all $e \in \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,*

f) $h_e\, \|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \pi_1^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\|_{0,e}^2$
$\le C_6 \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T}^2 + h_e\, \|\lambda - \lambda_h\|_{0,e}^2 + h_e\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}$,
for all $e \in \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge,

g) $h_e\, \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \le C_7 \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,T}^2 + h_T^2\, |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 + h_e\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}$
for all $e \in \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge,

h) $\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\, \|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\|_{0,e}^2 \le C_8 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \right\}$,
where, given $e \in \mathcal{E}_h(\Sigma)$, T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge.

i) $\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\mathbf{t} + \boldsymbol{\varphi}'_h \right\|^2_{0,e}$

$\displaystyle\le C_9 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|^2_{0,T} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{1/2,\Sigma} \right\},$

where given $e \in \mathcal{E}_h(\Sigma)$, T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. For a) and b) we refer to [13, Lemma 6.1]. Alternatively, a) and b) also follow from straightforward applications of the technical result provided in [6, Lemma 4.3] (see also [33, Lemma 4.9]). Similarly, for c) we refer to [13, Lemma 6.3] (see also [33, Lemma 4.13] or [25, Lemma 5.5]). On the other hand, the estimate given by d) corresponds to [5, Lemma 4.12]. Next, for e), f) and g) we refer to [36, Lemmas 3.15, 3.16 and 3.17]. The proofs of h) and i) follow from very slight modifications of the proof of [25, Lemma 5.7]. Alternatively, an *elasticity version* of h) and i), which is provided in [30, Lemma 20], can also be adapted to our case. □

Observe that the estimates h) and i) in the previous lemma are the only ones providing non-local bounds. However, under additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can give the following local bounds instead.

Lemma 5.14 *Assume that $\lambda|_e \in H^1(e)$, and $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$, for each $e \in \mathcal{E}_h(\Sigma)$. Then there exist C_1, $C_2 > 0$, such that for each $e \in \mathcal{E}_h(\Sigma)$ there hold*

$$h_e \left\|\mathrm{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda'_h\right\|^2_{0,e} \le C_1 \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T_e} + h_e \left\|\lambda' - \lambda'_h\right\|^2_{0,e} \right\},$$

and

$$h_e \left\| \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\mathbf{t} + \boldsymbol{\gamma}_{\mathrm{S},h}\mathbf{t} + \boldsymbol{\varphi}'_h \right\|^2_{0,e} \le C_2 \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T_e} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|^2_{0,T} + h_e \left\|\boldsymbol{\varphi}' - \boldsymbol{\varphi}'_h\right\|^2_{0,e} \right\}.$$

Proof. Similarly as for h) and i) in Lemma 5.13, these estimates follow by adapting the corresponding *elasticity version* in [30]. We omit further details. □

The following lemma is a direct consequence of [13, Lemma 6.2] (see also [6, Lemma 4.4]).

Lemma 5.15 *There exist positive constants C_i, $i \in \{1, ..., 4\}$, independent of h, such that*

a) $h_e \left\|[\mathrm{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\right\|^2_{0,e} \le C_1 \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,w_e}$ *for all $e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$, where the set w_e is given by $w_e := \cup\left\{T' \in \mathcal{T}_h^{\mathrm{D}} : e \in \mathcal{E}(T')\right\}$,*

b) $h_e \left\|\mathrm{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}\right\|^2_{0,e} \le C_2 \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^2_{0,T}$ *for all $e \in \mathcal{E}_h(\Gamma_{\mathrm{D}})$, where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,*

c) $h_e \left\|\left[\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\mathbf{t}\right]\right\|^2_{0,e} \le C_3 \left\{\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,w_e} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|^2_{0,w_e}\right\}$ *for all $e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$, where the set w_e is given by $w_e := \cup\left\{T' \in \mathcal{T}_h^{\mathrm{S}} : e \in \mathcal{E}(T')\right\}$,*

d) $h_e \left\|\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S},h}\right)\mathbf{t}\right\|^2_{0,e} \le C_4 \left\{\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|^2_{0,T}\right\}$ *for all $e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$, where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.*

We end this section by observing that the required efficiency of the a posteriori error estimator Θ follows straightforwardly from Lemmas 5.11, 5.12, 5.13, 5.14, and 5.15. In particular, the terms $h_e \|\lambda - \lambda_h\|_{0,e}^2$ and $h_e \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2$, in Lemma 5.13 (d), e), f) and g)), are bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\lambda - \lambda_h\|_{0,e}^2 \,\leq\, h \|\lambda - \lambda_h\|_{0,\Sigma}^2 \,\leq\, C\, h \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 ,$$

and

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, h \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \,\leq\, C\, h \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}^2 .$$

6 Numerical results

We now turn to the implementation of a few numerical tests that confirm the predicted features of the two proposed schemes, including optimal convergence rates, reliability and efficiency of the associated a posteriori error estimators, and adaptive mesh refinement. As usual, N_h stands for the total number of degrees of freedom of a given scheme, individual and total errors in the natural norms are defined as

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) = \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\mathbf{u}_{\mathrm{D}}) = \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_{\mathrm{D}}}, \quad \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) = \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} ,$$

$$\mathsf{e}(\boldsymbol{\varphi}) = \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}, \qquad \mathsf{e}(\lambda) = \|\lambda - \lambda_h\|_{1/2,\Sigma} ,$$

$$\mathsf{e}_i(\mathbf{u}_{\mathrm{S}}) = \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{i,\Omega_{\mathrm{S}}} \quad \forall\, i \in \{0,1\} , \qquad \mathsf{e}(p_{\mathrm{D}}) = \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} ,$$

$$\mathsf{e} = \Big\{[\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})]^2 + [\mathsf{e}(\mathbf{u}_{\mathrm{D}})]^2 + [\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})]^2 + [\mathsf{e}(\boldsymbol{\varphi})]^2 + [\mathsf{e}(\lambda)]^2 + [\mathsf{e}_i(\mathbf{u}_{\mathrm{S}})]^2 + [\mathsf{e}(p_{\mathrm{D}})]^2 + |\mu|^2\Big\}^{1/2} ,$$

and the effectivity index associated to the indicator Θ is $\mathtt{eff}(\Theta) = \mathsf{e}/\Theta$. Rates of convergence associated to quasi-uniform and adaptive refinements of a mesh are given, respectively, by

$$r^{qu}(\cdot) := \frac{\log(\mathsf{e}(\cdot)/\widehat{\mathsf{e}}(\cdot))}{\log(h/\widehat{h})}, \quad r^{a}(\cdot) := \frac{\log(\mathsf{e}(\cdot)/\widehat{\mathsf{e}}(\cdot))}{-\frac{1}{2}\log(N_h/N_{\widehat{h}})},$$

where e and $\widehat{\mathsf{e}}$ denote errors computed on two consecutive meshes of sizes h and $\widehat{h}$, with N_h and $N_{\widehat{h}}$ degrees of freedom, respectively.

6.1 Example 1: accuracy of the augmented finite element formulation

Our first example consists of a porous unit square, coupled with a semi-disk-shaped fluid domain, i.e., $\Omega_{\mathrm{D}} = (0,1)^2$ and $\Omega_{\mathrm{S}} = \{(x_1,x_2) : (x_1 - 1/2)^2 + (x_2-1)^2 < 1,\ x_2 > 1\}$ (see bottom left panel of Figure 6.1). In these domains, we adequately manufacture the data in (3.9) and (4.4) so that a smooth exact solution in the tombstone-shaped domain Ω is given by

$$\begin{aligned}
&\boldsymbol{\sigma}_{\mathrm{S}} = K\cos(\pi x_1)\cos(\pi x_2)\begin{pmatrix} -2\nu\pi^2 - 1 & 0 \\ 0 & 2\nu\pi^2 - 1\end{pmatrix}, \ \boldsymbol{\gamma}_{\mathrm{S}} = \frac{K\pi^2}{2}\sin(\pi x_1)\sin(\pi x_2)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \\
&\mathbf{u}_{\mathrm{S}} = \begin{pmatrix} -K\pi\sin(\pi x_1)\cos(\pi x_2) \\ K\pi\cos(\pi x_1)\sin(\pi x_2)\end{pmatrix}, \ \mathbf{u}_{\mathrm{D}} = \begin{pmatrix} K\pi\sin(\pi x_1)\cos(\pi x_2) \\ K\pi\cos(\pi x_1)\sin(\pi x_2)\end{pmatrix}, \ p_{\mathrm{D}} = K\cos(\pi x_1)\cos(\pi x_2).
\end{aligned} \tag{6.1}$$

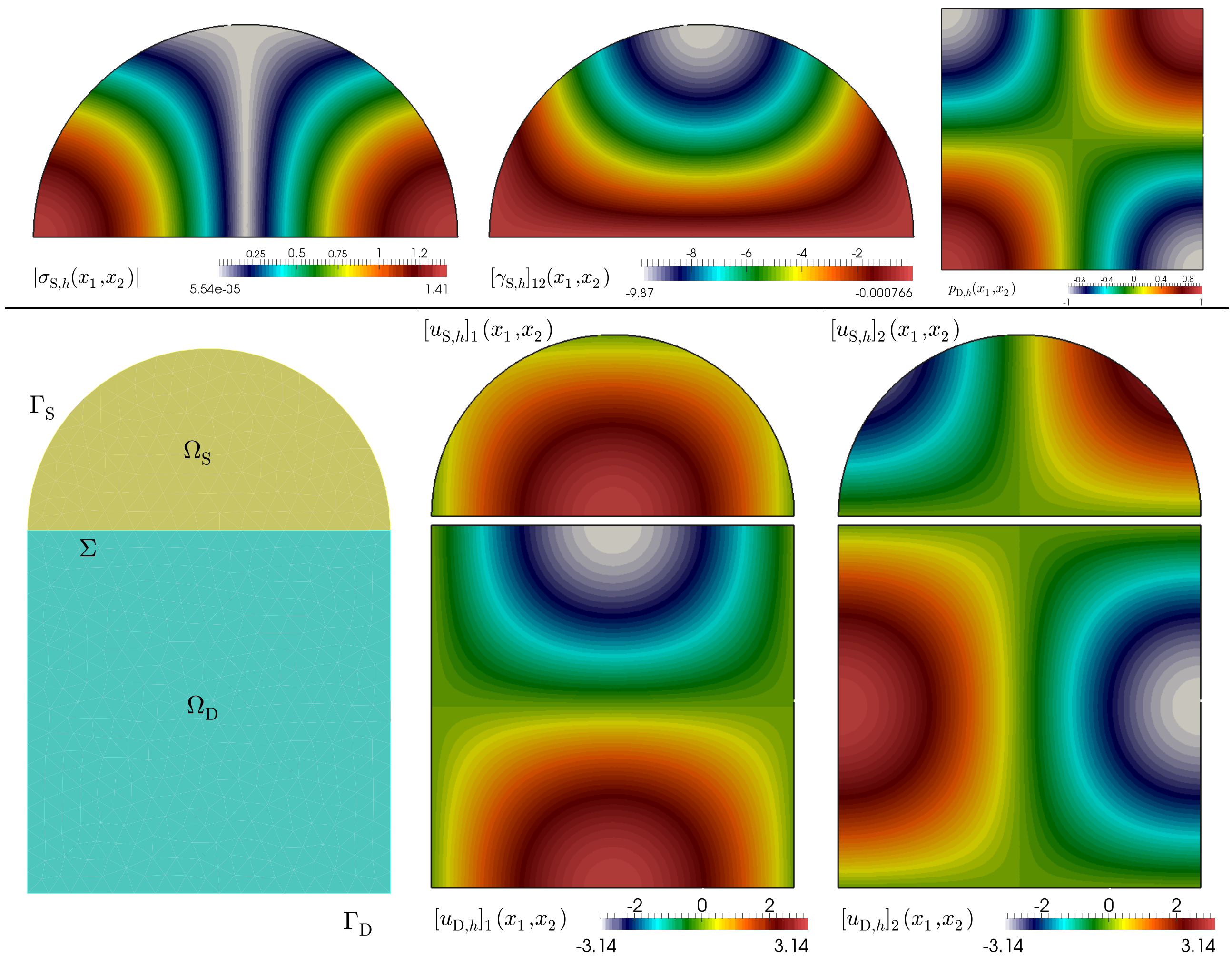

Figure 6.1: Example 1: top: approximated spectral norm of the Cauchy stress tensor, component (1,2) of the skew-symmetric part of the Stokes velocity gradient, and Darcy pressure field (left, middle and right, respectively). Bottom: geometry configuration (left) and velocity components on the whole domain (middle and right, respectively). For the latter, the Stokes domain is shifted by $\delta x_2 = 0.025$ for visualization purposes only. Grids of 98364 and 97452 were employed for the discretization of Ω_S and Ω_D, respectively.

Notice that this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ, and the the boundary condition $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D. However the Dirichlet condition for the Stokes velocity on Γ_S is non-homogeneous ($\mathbf{u}_S = \mathbf{g}$, with $\mathbf{g}$ as in (6.1)), which implies that the linear functionals $\mathcal{F}$ and $\mathbf{F}$ defined in (3.10) and (4.5), respectively, exhibit an extra term $\langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S}$. In Figure 6.1 we depict the approximate solutions obtained with the augmented formulation from Section 4.3.1. Model and stabilization parameters were set as $\nu = 0.001$, $K = 1$, $\pi_1 = 100K/\nu$, $\kappa_1 = \nu$, $\kappa_2 = 2\nu$, $\kappa_3 = \nu\kappa_2$. We generate an initial unstructured mesh for Ω and apply several refinement and smoothing steps to measure errors in different norms. These are displayed in Table 6.1, where we can observe a first order convergence for all fields, confirming the expected results from Theorem 4.6. In addition, we observe an effectivity index which remains bounded and oscillation free independently of the refinement level. This behavior illustrates the reliability and efficiency of Θ in the case of smooth solutions.

N_h	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r^{qu}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$r^{qu}(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r^{qu}(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\varphi})$	$r^{qu}(\boldsymbol{\varphi})$
159	0.446196	–	1.580782	–	2.603010	–	0.205070	–
257	0.226722	0.943739	0.875395	0.842095	1.426422	0.789102	0.101241	0.868981
503	0.118823	0.981413	0.448785	0.956574	0.720729	0.896044	0.047451	1.023749
1217	0.052284	0.964844	0.212825	0.972366	0.362127	0.969155	0.024874	0.897510
3725	0.027158	0.982517	0.123946	1.015239	0.180258	1.023364	0.012634	0.984509
13051	0.013688	1.054410	0.066940	0.948094	0.094218	0.982391	0.005881	1.043483
48606	0.007136	1.014553	0.039522	0.820726	0.045701	1.039798	0.002955	0.895183
188441	0.003885	0.897878	0.019889	0.929132	0.029973	0.926577	0.001562	0.972879
579852	0.001975	0.914762	0.010587	0.916879	0.014799	0.978820	0.007868	0.934747
N_h	$\mathtt{e}(\lambda)$	$r^{qu}(\lambda)$	$\mathtt{e}_1(\mathbf{u}_{\mathrm{S}})$	$r^{qu}(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(p_{\mathrm{D}})$	$r^{qu}(p_{\mathrm{D}})$		$\mathtt{eff}(\Theta)$
159	1.290934	–	0.958703	–	0.447035	–		0.412430
257	0.694226	0.920943	0.475323	0.878973	0.224808	0.953157		0.418103
503	0.357180	0.931064	0.252024	0.928001	0.113783	0.870415		0.425560
1217	0.176107	0.986271	0.131323	0.891469	0.061505	0.946765		0.420781
3725	0.098482	0.869063	0.061902	1.015167	0.030388	0.874452		0.413923
13051	0.050354	0.918621	0.031949	0.977998	0.015127	0.950913		0.415893
48606	0.025166	0.955871	0.018239	0.873081	0.008851	0.834767		0.418906
188441	0.013573	0.935901	0.009178	0.932855	0.004364	1.004215		0.420651
579852	0.006729	0.940327	0.004739	0.943360	0.002578	0.936840		0.406982

Table 6.1: Example 1: convergence tests against analytical solutions employing the augmented finite element formulation on a sequence of quasi-uniformly refined triangulations of the tombstone-shaped domain.

6.2 Example 2: a posteriori error estimation and mesh adaptation

Next, we assess the reliability and efficiency of the proposed a posteriori error estimators applied to the augmented discretization of the coupled problem defined on the inversed-L-shaped domain $\Omega = \Omega_{\mathrm{D}} \cup \Omega_{\mathrm{S}}$, where $\Omega_{\mathrm{S}} = (0,1)^2$ and $\Omega_{\mathrm{D}} = (-1,1)\times(-1,0)$, representing a fluid channel on top of a porous basin. We compute errors between approximate solutions and the following exact solutions:

$$
\begin{aligned}
(\boldsymbol{\sigma}_{\mathrm{S}})^{\mathtt{t}}_{1,\cdot} &= \begin{pmatrix} -x_1(x_1-1)x_2(x_2-1) - 512\nu x_1(x_1-1)(2x_1-1)x_2(x_2-1)(2x_2-1) \\ 128\nu(x_2^2(x_2-1)^2(6x_1^2-6x_1+1) - x_1^2(x_1-1)^2(6x_2^2-6x_2+1)) \end{pmatrix}, \\
(\boldsymbol{\sigma}_{\mathrm{S}})^{\mathtt{t}}_{2,\cdot} &= \begin{pmatrix} 128\nu(x_2^2(x_2-1)^2(6x_1^2-6x_1+1) - x_1^2(x_1-1)^2(6x_2^2-6x_2+1)) \\ -x_1(x_1-1)x_2(x_2-1) + 512\nu x_1(x_1-1)(2x_1-1)x_2(x_2-1)(2x_2-1) \end{pmatrix}, \\
\boldsymbol{\gamma}_{\mathrm{S}} &= -128(x_2^2(x_2-1)^2(6x_1^2-6x_1+1) + x_1^2(x_1-1)^2(6x_2^2-6x_2+1)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\mathbf{u}_{\mathrm{S}} &= \begin{pmatrix} -128x_1^2(x_1-1)^2x_2(x_2-1)(2x_2-1) \\ 128x_2^2(x_2-1)^2x_1(x_1-1)(2x_1-1) \end{pmatrix}, \; p_{\mathrm{D}} = \frac{(x_1-1)^2(x_1+1)^2x_2^2(x_2+1)^2}{8r(x_1,x_2)}, \\
\mathbf{u}_{\mathrm{D}} &= \begin{pmatrix} \dfrac{K(x_1-x_a)(x_1-1)^2(x_1+1)^2x_2^2(x_2+1)^2}{4r(x_1,x_2)^2} - \dfrac{Kx_1(x_1-1)(x_1+1)x_2^2(x_2+1)^2}{2r(x_1,x_2)} \\ \dfrac{K(x_2-x_b)(x_1-1)^2(x_1+1)^2x_2^2(x_2+1)^2}{4r(x_1,x_2)^2} - \dfrac{K(x_1-1)^2(x_1+1)^2x_2(x_2+1)(2x_2+1)}{4r(x_1,x_2)} \end{pmatrix},
\end{aligned}
\tag{6.2}
$$

N_h	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r^{qu}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$r^{qu}(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r^{qu}(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\varphi})$	$r^{qu}(\boldsymbol{\varphi})$
72	0.040405	–	0.588039	–	4.668901	–	1.958742	–
154	0.029925	0.473399	0.298587	0.913151	3.981857	0.219451	1.528710	0.378051
499	0.020034	0.573736	0.176233	0.753912	2.026571	0.930233	0.943701	0.660789
1713	0.010613	0.924945	0.084169	0.933307	1.015841	0.957168	0.623376	0.741120
6183	0.007432	0.540744	0.043101	0.911502	0.567462	0.885169	0.362061	0.831810
23329	0.004695	0.469161	0.022937	0.914447	0.347704	0.753473	0.203162	0.828498
91509	0.002865	0.510538	0.012809	0.832579	0.203847	0.838172	0.102250	0.920981
361596	0.001911	0.612654	0.008457	0.854137	0.138111	0.861485	0.061773	0.811624

N_h	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r^{a}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$r^{a}(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r^{a}(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\varphi})$	$r^{a}(\boldsymbol{\varphi})$
130	0.039976	–	0.150448	–	2.963713	–	1.428789	–
291	0.029859	0.989201	0.072126	0.948069	1.040652	2.319315	0.578691	1.093129
835	0.016833	0.959438	0.029135	0.952439	0.483704	1.453607	0.152965	0.944469
2730	0.009548	0.970485	0.010790	0.975378	0.214326	0.961382	0.054455	0.995687
9297	0.003303	0.989528	0.003880	1.013111	0.070319	0.983268	0.018415	0.947479
20037	0.001058	0.993040	0.002089	1.012786	0.037719	0.970788	0.008739	0.960034
73006	0.000332	0.992797	0.000719	1.009012	0.012658	1.000334	0.002616	0.937121
450405	0.000051	0.984944	0.000186	1.008422	0.002587	1.098927	0.000354	0.959526

N_h	$\mathtt{e}(\lambda)$	$r^{qu}(\lambda)$	$\mathtt{e}_1(\mathbf{u}_{\mathrm{S}})$	$r^{qu}(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(p_{\mathrm{D}})$	$r^{qu}(p_{\mathrm{D}})$		$\mathtt{eff}(\Theta)$
72	0.950747	–	1.126320	–	1.151813	–		0.462718
154	0.658887	0.608660	0.604480	0.880763	0.897591	0.696676		0.524565
499	0.325962	1.058434	0.549237	0.137038	0.448996	0.985259		0.646848
1713	0.168479	1.000822	0.297921	0.890480	0.240552	0.775353		0.589838
6183	0.099579	0.678598	0.164069	0.905367	0.128657	0.972616		0.418518
23329	0.068100	0.663121	0.103036	0.753929	0.068653	0.900161		0.570318
91509	0.043604	0.741581	0.069417	0.569783	0.038347	0.811470		0.462431
361596	0.032067	0.554231	0.053627	0.389962	0.021209	0.875454		0.396597

N_h	$\mathtt{e}(\lambda)$	$r^{a}(\lambda)$	$\mathtt{e}_1(\mathbf{u}_{\mathrm{S}})$	$r^{a}(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(p_{\mathrm{D}})$	$r^{a}(p_{\mathrm{D}})$		$\mathtt{eff}(\Theta)$
130	0.523492	–	0.818786	–	0.400247	–		0.286920
291	0.065701	0.510267	0.532226	0.877326	0.232092	0.985890		0.275054
835	0.034862	0.943771	0.239604	0.956598	0.072727	0.971858		0.280720
2730	0.016948	0.989210	0.088538	0.941544	0.024475	0.951104		0.279273
9297	0.006101	0.954129	0.027505	0.972996	0.008589	0.946897		0.272122
20037	0.003127	0.970293	0.016035	0.983535	0.005021	0.972749		0.281801
73006	0.001090	0.963957	0.005632	0.982031	0.001727	0.950892		0.279262
450405	0.000283	0.970045	0.001696	0.952306	0.000513	0.965440		0.279485

Table 6.2: Example 2: Convergence tests against analytical solutions employing the augmented formulation on a sequence of quasi-uniformly and adaptively refined triangulations of the inversed-L-shaped domain.

where $x_a = -\frac{1}{20}, x_b = \frac{1}{20}$ and $r = (x_1 - x_a)^2 + (x_2 - x_b)^2$. Notice that the Darcy velocity and pressure exhibit high gradients near the origin. The solutions in (6.2) satisfy all boundary conditions assumed in Section 2. We set numerical and physical parameters as follows: $\nu = 0.01$, $K = 1$, $\pi_1 = 1 \times 10^{-4}$.

Table 6.2 summarizes the convergence history of the method applied to a sequence of quasi-uniformly and adaptively refined triangulations of the domain. From the first rows, and in contrast to what was observed in the previous example, we notice that the lack of regularity of the exact solutions and the nonconvexity of the domain yield hindered convergence rates for practically all fields, along with an oscillation of the effectivity index. These problems are ammended after applying a classical adaptive mesh refinement procedure based on the estimator Θ (here we employ the so-called

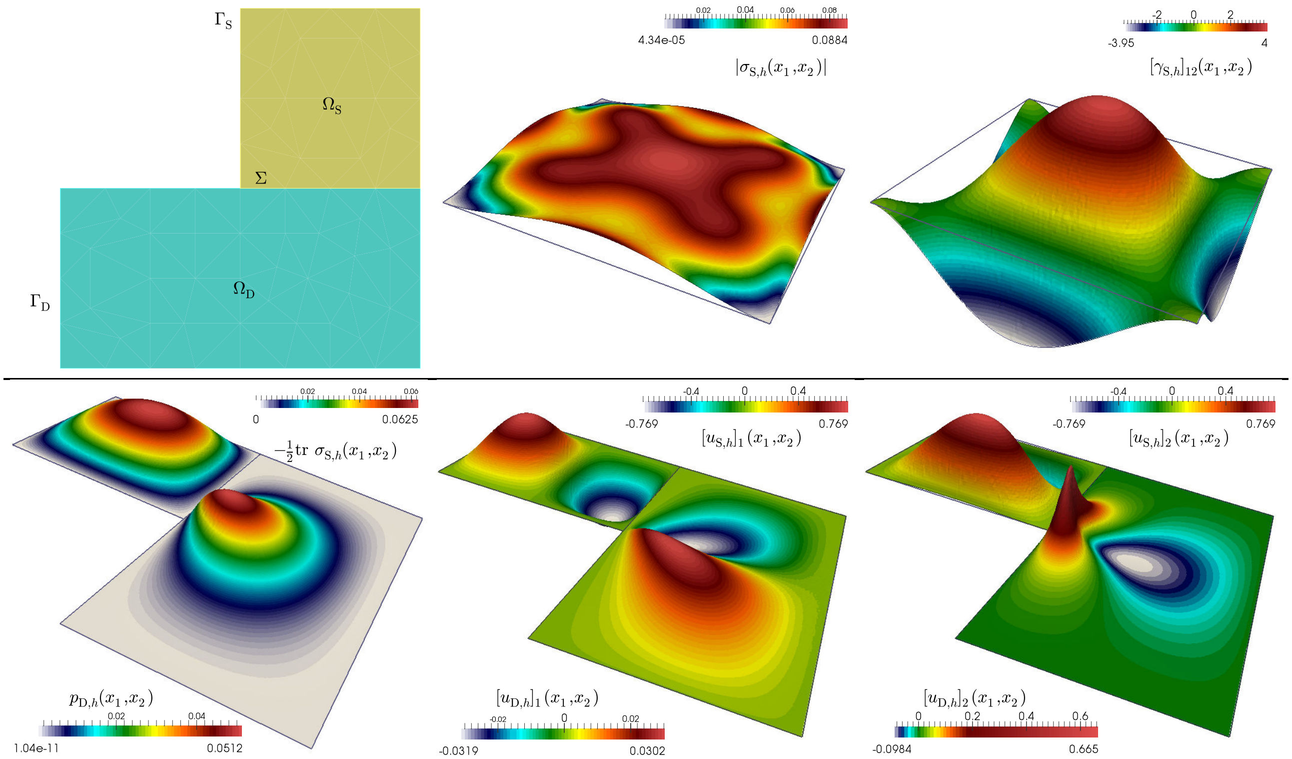

Figure 6.2: Example 2: top: geometry configuration, approximated spectral norm of the Cauchy stress tensor, and component (1,2) of the skew-symmetric part of the Stokes velocity gradient (left, middle and right, respectively). Bottom: Darcy pressure field, (left) and velocity components on the whole domain (middle and right, respectively). Approximate solutions were obtained with the augmented formulation.

blue-green algorithm. For its details we refer to e.g. [49]). The optimal convergence rates predicted by Theorem 3.8 are recovered (see bottom rows in Table 6.2). In addition, we provide in Figure 6.3 snapshots of intermediate steps of the adaptive algorithm, where it is clear that the indicator identifies well the regions of high gradients.

6.3 Example 3: flow in a porous medium with a vertical crack

Our following example focuses on the simulation of flow in a porous medium with a vertical crack of thickness equal to 0.1, similar to the ones presented in [8, Section 7.1]. In this case, the flow domain is the rectangle $\Omega_{\mathrm{S}} = [-0.05, 0.05] \times [-0.4, 0]$ representing a crack emerging to the surface ($x_2 = 0$), and the porous domain is the octagon $\Omega_{\mathrm{D}} = [-1,1]\times[-1,0]\setminus\Omega_{\mathrm{S}}$, so $\Omega = [-1,1]\times[-1,0]$ and $\Sigma = \partial\Omega \setminus ([-0.05, 0.05] \times \{0\})$ (see Figure 6.4, bottom right plot). Values for viscosity and porosity correspond to the case of water flowing in calcarenite mixed with sand, i.e., $\nu = 0.01, K = 0.001$ and we set $\pi_1 = 100K/\nu$, $\kappa_1 = \nu$, $\kappa_2 = 2\nu$, $\kappa_3 = 0.01\kappa_2$. The external forces correspond to gravity $\mathbf{f}_{\mathrm{S}} = (0,-1)^{\mathtt{t}}$, and a non-homogeneous slip boundary condition is imposed on Γ_{D}: $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = (0,-1)^{\mathtt{t}}\cdot\mathbf{n}$, representing a rainfall rate. Figure 6.4 depicts the approximate solutions, matching satisfactorily the results from [8]. Here we have employed the fully mixed formulation proposed in Section 3.4.1.

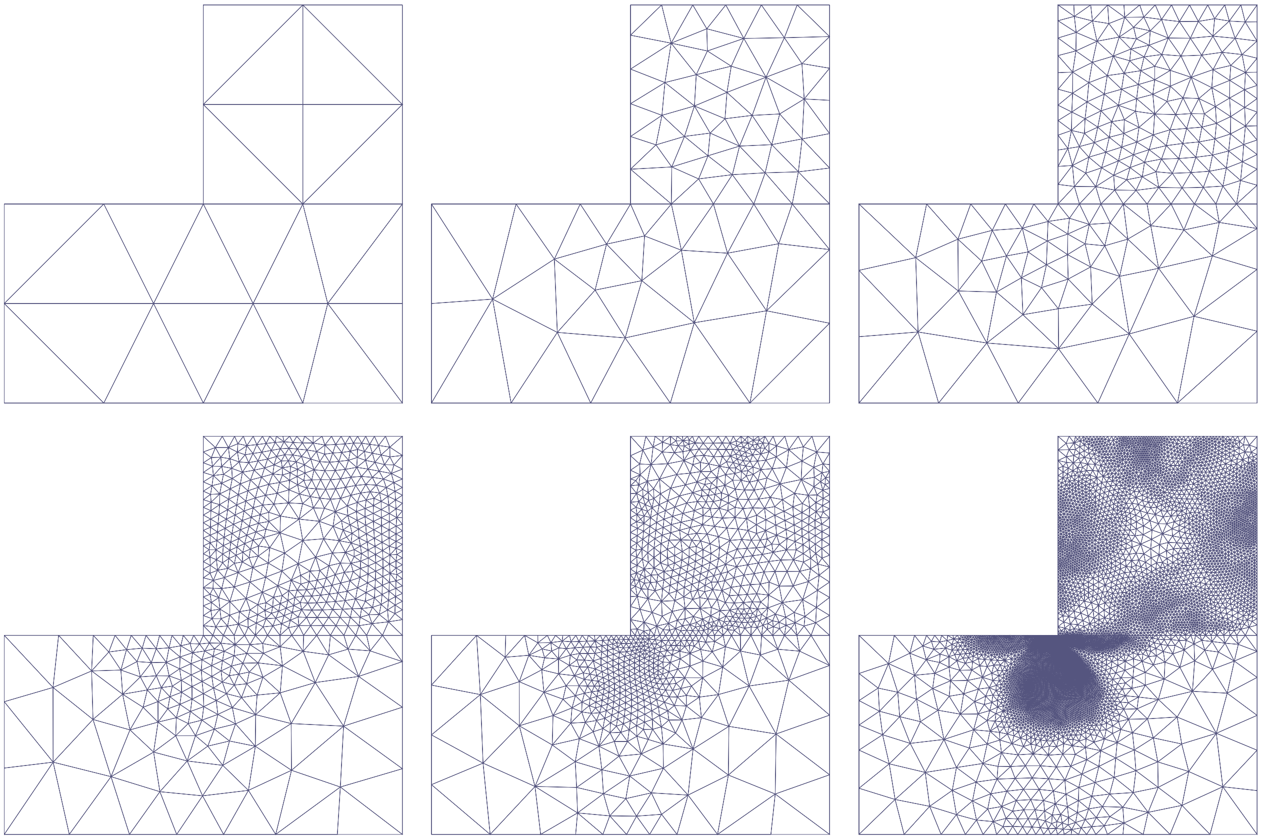

Figure 6.3: Example 2: meshes adaptively refined using the indicator Θ defined in (5.3).

6.4 Example 4: a porous sphere immersed in a fluid cavity

With the aim of testing the 3-D implementation of the proposed augmented formulation and particular finite element family specified in Section 4.3.2, we close this section with a simple simulation of a three-dimensional porous domain fully immersed in a cubic fluid domain. We stress that even though the continuous and discrete analysis of such a formulation was not included in the present study, similar tools to those employed in [35] can be applied to extend the present framework and cover this case. We consider the domains $\Omega = (-\frac{1}{2}, \frac{1}{2})^3$, $\Omega_{\mathrm{D}} = \{(x_1, x_2, x_3) : \ x_1^2 + x_2^2 + x_3^2 \leq \frac{1}{25}\}$ and $\Omega_{\mathrm{S}} = \Omega \setminus \Omega_{\mathrm{D}}$ and construct a tetrahedral mesh of 4575 vertices and 49578 elements to discretize Ω_{S}, whereas the grid for Ω_{D} consists of 12843 vertices and 61450 tetrahedra. The model and stabilization parameters are set as $\nu = 1$, $K = 100$, $\pi_1 = \nu/K$, $\pi_2 = \pi_1^{-1}$, $\kappa_1 = \nu$, $\kappa_2 = 2\nu$, and $\kappa_3 = \kappa_2/K$, and we impose the following forcing terms

$$\mathbf{f}_{\mathrm{S}} = (0, 0, \sin(\pi(x_3 - 0.5)))^{\mathtt{t}}, \quad f_{\mathrm{D}} = K^{-1} \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3).$$

Approximate solutions are reported in Figure 6.5.

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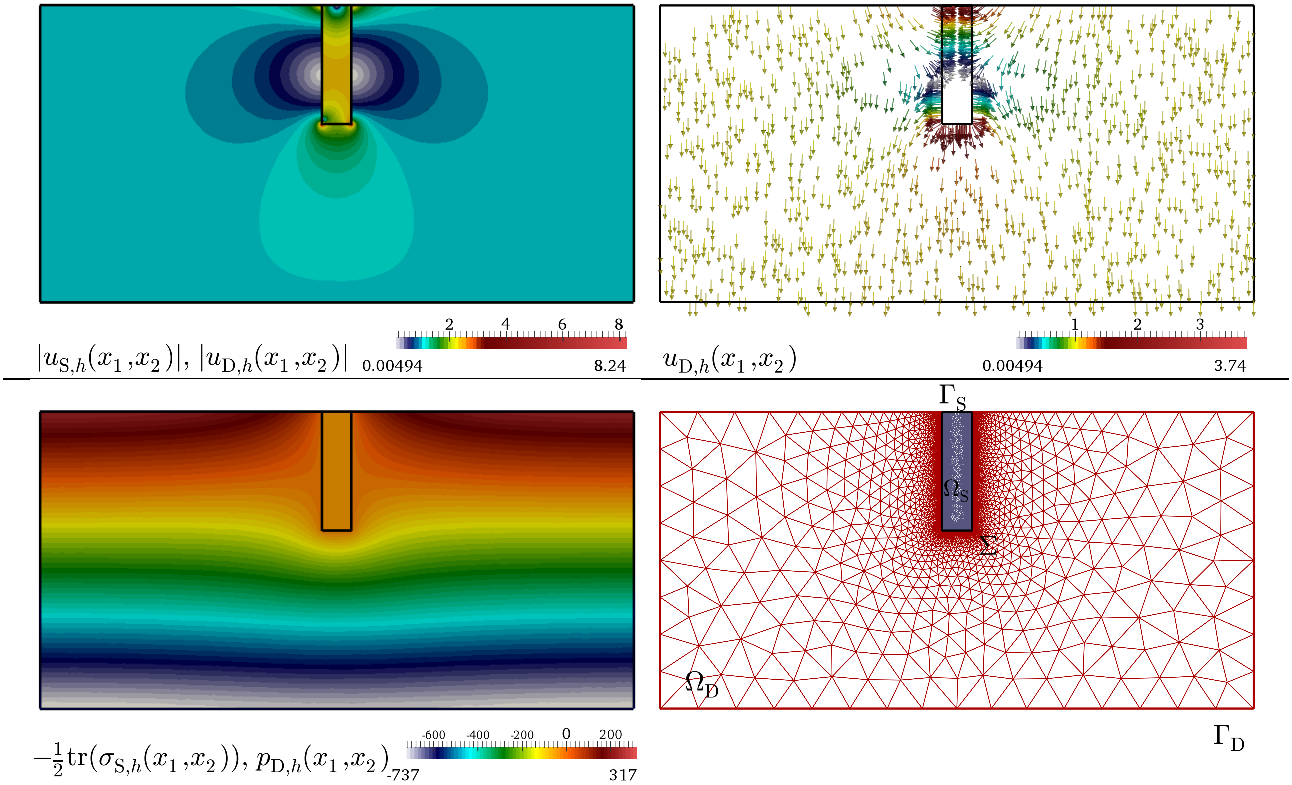

Figure 6.4: Example 3: Darcy and Stokes approximate velocity magnitude (top left), approximated Darcy velocity vectors (top right panel), computed Darcy pressure profile and postprocessed Stokes pressure (bottom left), and configuration of geometries and meshes (bottom right) for the Stokes-Darcy coupling modeling the flow of water into a mixture of calcarenite and sand.

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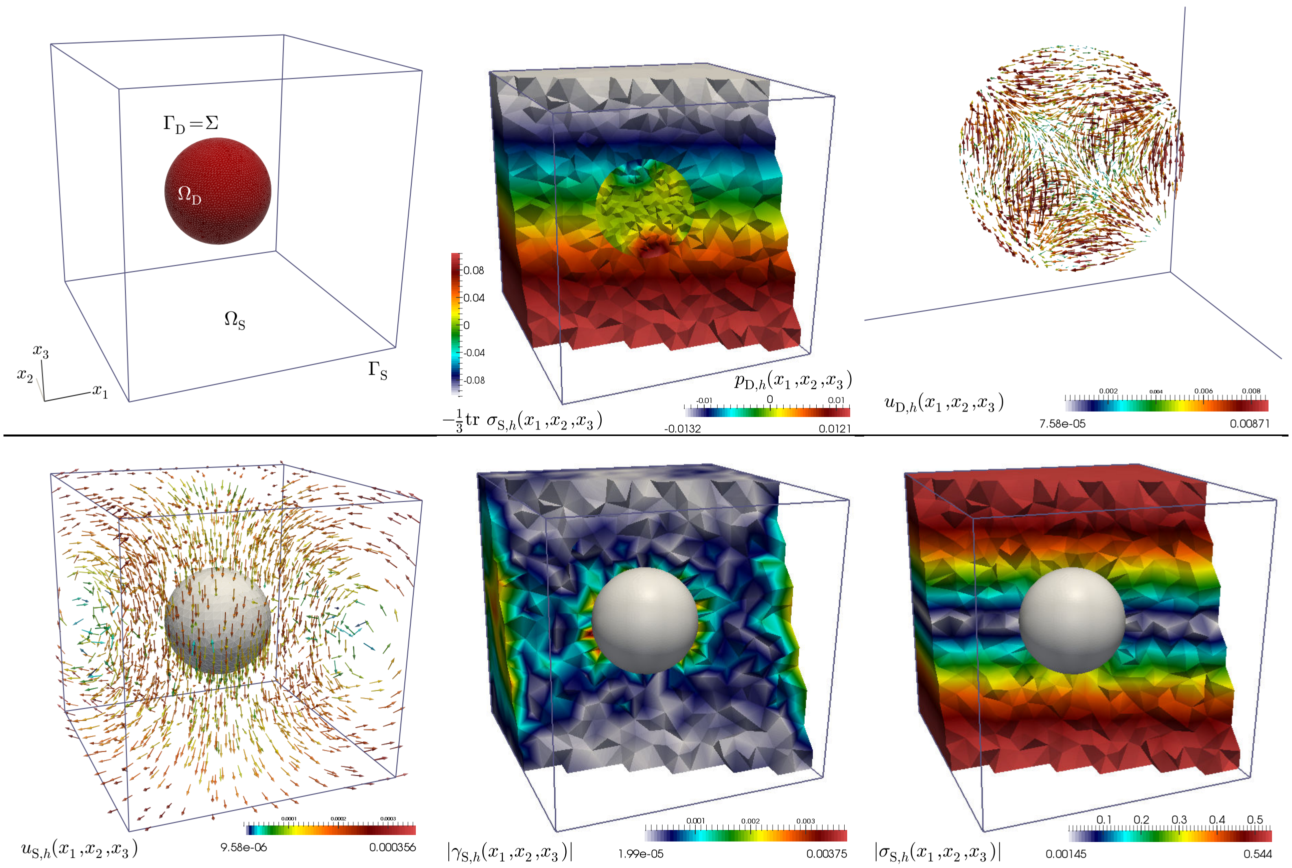

Figure 6.5: Example 4. Augmented mixed finite element approximation of the flow in a cubic cavity containing a spheric obstacle. Top: domain and boundaries configuration (left), crinkle clip of the Stokes and Darcy pressure fields (center), and zoom of the Darcy velocity vectors (right). Bottom: Stokes velocity vectors (left), clip of the magnitude of the skew-symmetric part of the Stokes velocity gradient (middle), and clipped magnitude of the Cauchy stress tensor (right).

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